2 LIMITS

2.1 The Limit Idea: Instantaneous Velocity and Tangent Lines

Preliminary Questions

1. Average velocity is equal to the slope of a secant line through two points on a graph. Which graph?

SOLUTION Average velocity is the slope of a secant line through two points on the graph of position as a function of time.

2. Can instantaneous velocity be defined as a ratio? If not, how is instantaneous velocity computed?

SOLUTION Instantaneous velocity cannot be defined as a ratio. It is defined as the limit of average velocity as time elapsed shrinks to zero.

3. With t in hours, at $t = 0$ Dale entered Highway 1. At $t = 2$ he was 126 miles down the highway, on the side of the road with a flat tire. At $t = 3$ he was still on the side of the road, waiting for road assistance. What was Dale's average velocity over each of the time intervals:

(a) From $t = 0$ to $t = 2$

(b) From $t = 0$ to $t = 3$

(c) From $t = 2$ to $t = 3$

SOLUTION

(a) Over the time interval from $t = 0$ to $t = 2$, Dale traveled 126 miles. His average velocity was therefore

$$\frac{126}{2-0} = 63 \text{ miles/hour}$$

(b) Over the time interval from $t = 0$ to $t = 3$, Dale traveled 126 miles. His average velocity was therefore

$$\frac{126}{3-0} = 42 \text{ miles/hour}$$

(c) Over the time interval from $t = 2$ to $t = 3$, Dale traveled 0 miles. His average velocity was therefore

$$\frac{0}{3-2} = 0 \text{ miles/hour}$$

4. What is the graphical interpretation of instantaneous velocity at a specific time $t = t_0$?

SOLUTION Instantaneous velocity at time $t = t_0$ is the slope of the line tangent to the graph of position as a function of time at $t = t_0$.

Exercises

1. A ball dropped from a state of rest at time $t = 0$ travels a distance $s(t) = 4.9t^2$ m in t seconds.

(a) How far does the ball travel during the time interval [2, 2.5]?

(b) Compute the average velocity over [2, 2.5].

(c) Compute the average velocity for the time intervals in the table and estimate the ball's instantaneous velocity at $t = 2$.

Interval	[2, 2.01]	[2, 2.005]	[2, 2.001]	[2, 2.00001]
Average velocity				

SOLUTION

(a) Given $s(t) = 4.9t^2$, the ball travels $\Delta s = s(2.5) - s(2) = 4.9(2.5)^2 - 4.9(2)^2 = 11.025$ m during the time interval [2, 2.5].

(b) The average velocity over [2, 2.5] is

$$\frac{\Delta s}{\Delta t} = \frac{s(2.5) - s(2)}{2.5 - 2} = \frac{11.025}{0.5} = 22.05 \text{ m/s}$$

(c)

time interval	[2, 2.01]	[2, 2.005]	[2, 2.001]	[2, 2.00001]
average velocity	19.649	19.6245	19.6049	19.600049

The instantaneous velocity at $t = 2$ is approximately 19.6 m/s.

2. A wrench dropped from a state of rest at time $t = 0$ travels a distance $s(t) = 4.9t^2$ m in t seconds. Estimate the instantaneous velocity at $t = 3$.

SOLUTION To estimate the instantaneous velocity, we compute the average velocities:

time interval	[3, 3.01]	[3, 3.005]	[3, 3.001]	[3, 3.00001]
average velocity	29.449	29.4245	29.4049	29.400049

The instantaneous velocity is approximately 29.4 m/s.

3. On her bicycle ride Fabiana's position (in km) as a function of time (in hours) is $s(t) = 22t + 17$. What was her average velocity between $t = 2$ and $t = 3$? What was her instantaneous velocity at $t = 2.5$?

SOLUTION Fabiana's average velocity between $t = 2$ and $t = 3$ was

$$\frac{s(3) - s(2)}{3 - 2} = \frac{83 - 61}{1} = 22 \text{ km/hour}$$

To estimate the instantaneous velocity, we compute the average velocities:

time interval	[2.5, 2.51]	[2.5, 2.501]	[2.5, 2.5001]	[2.5, 2.50001]
average velocity	22	22	22	22

The instantaneous velocity is 22 km/hour.

4. Compute $\Delta y/\Delta x$ for the interval [2, 5], where $y = 4x - 9$. What is the slope of the tangent line at $x = 2$?

SOLUTION $\Delta y/\Delta x = ((4(5) - 9) - (4(2) - 9))/(5 - 2) = 4$. Because the graph of $y = 4x - 9$ is a line, it is the tangent line to the graph for all x. And since the line has slope 4, that is the slope of the tangent line at $x = 2$.

In Exercises 5–6, a ball is dropped on Mars where the distance traveled is $s(t) = 1.9t^2$ *meters in t seconds.*

5. Compute the ball's average velocity over the time interval [3, 6] and estimate the instantaneous velocity at $t = 3$.

SOLUTION The ball's average velocity over the time interval [3, 6] is

$$\frac{s(6) - s(3)}{6 - 3} = \frac{68.4 - 17.1}{3} = 17.1 \text{ m/s}$$

To estimate the instantaneous velocity, we compute the average velocities:

time interval	[3, 3.1]	[3, 3.01]	[3, 3.001]	[3, 3.0001]
average velocity	11.59	11.419	11.4019	11.40019

The instantaneous velocity is approximately 11.4 m/s.

6. Compute the ball's average velocity over the time interval [5, 9] and estimate the instantaneous velocity at $t = 5$.

SOLUTION The ball's average velocity over the time interval [5, 9] is

$$\frac{s(9) - s(5)}{9 - 5} = \frac{153.9 - 47.5}{4} = 26.6 \text{ m/s}$$

To estimate the instantaneous velocity, we compute the average velocities:

time interval	[5, 5.1]	[5, 5.01]	[5, 5.001]	[5, 5.0001]
average velocity	19.19	19.019	19.0019	19.00019

The instantaneous velocity is approximately 19.0 m/s.

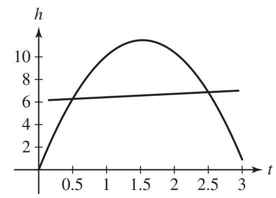
In Exercises 7–8, a stone is tossed vertically into the air from ground level with an initial velocity of 15 m/s. *Its height at time t is* $h(t) = 15t - 4.9t^2$ m.

7. Compute the stone's average velocity over the time interval [0.5, 2.5] and indicate the corresponding secant line on a sketch of the graph of h.

SOLUTION The average velocity is equal to

$$\frac{h(2.5) - h(0.5)}{2} = 0.3 \text{ m/s}$$

The secant line is plotted with $h(t)$ below.



8. Compute the stone's average velocity over the time intervals [1, 1.01], [1, 1.001], [1, 1.0001] and [0.99, 1], [0.999, 1], [0.9999, 1], and then estimate the instantaneous velocity at $t = 1$.

SOLUTION With $h(t) = 15t - 4.9t^2$, the average velocity over the time interval $[t_1, t_2]$ is given by

$$\frac{\Delta h}{\Delta t} = \frac{h(t_2) - h(t_1)}{t_2 - t_1}$$

time interval	[1, 1.01]	[1, 1.001]	[1, 1.0001]	[0.99, 1]	[0.999, 1]	[0.9999, 1]
average velocity	5.151	5.1951	5.1995	5.249	5.2049	5.2005

The instantaneous velocity at $t = 1$ second is 5.2 m/s.

9. The position of a particle at time t is $s(t) = 2t^3$. Compute the average velocity over the time interval [2, 4] and estimate the instantaneous velocity at $t = 2$.

SOLUTION The average velocity over the time interval [2, 4] is

$$\frac{s(4) - s(2)}{4 - 2} = \frac{128 - 16}{2} = 56$$

To estimate the instantaneous velocity at $t = 2$, we examine the following table.

time interval	[2, 2.01]	[2, 2.001]	[2, 2.0001]	[1.99, 2]	[1.999, 2]	[1.9999, 2]
average velocity	24.1202	24.012	24.0012	23.8802	23.988	23.9988

The instantaneous velocity at $t = 2$ is approximately 24.0.

10. The position of a particle at time t is $s(t) = t^3 + t$. Compute the average velocity over the time interval [1, 4] and estimate the instantaneous velocity at $t = 1$.

SOLUTION The average velocity over the time interval [1, 4] is

$$\frac{s(4) - s(1)}{4 - 1} = \frac{68 - 2}{3} = 22$$

To estimate the instantaneous velocity at $t = 1$, we examine the following table.

time interval	[1, 1.01]	[1, 1.001]	[1, 1.0001]	[0.99, 1]	[0.999, 1]	[0.9999, 1]
average velocity	4.0301	4.0030	4.0003	3.9701	3.9970	3.9997

The instantaneous velocity at $t = 1$ is approximately 4.0.

In Exercises 11–18, estimate the slope of the tangent line at the point indicated.

11. $f(x) = x^2 + x; \quad x = 0$

SOLUTION

x interval	[0, 0.01]	[0, 0.001]	[0, 0.0001]	[−0.01, 0]	[−0.001, 0]	[−0.0001, 0]
slope of secant	1.01	1.001	1.0001	0.99	0.999	0.9999

The slope of the tangent line at $x = 0$ is approximately 1.0.

12. $P(x) = 3x^2 - 5$; $x = 2$

SOLUTION

x interval	[2, 2.01]	[2, 2.001]	[2, 2.0001]	[1.99, 2]	[1.999, 2]	[1.9999, 2]
slope of secant	12.03	12.003	12.0003	11.97	11.997	11.9997

The slope of the tangent line at $x = 2$ is approximately 12.0.

13. $f(t) = 12t - 7$; $t = -4$

SOLUTION

t interval	[−4, −3.99]	[−4, −3.999]	[−4, −3.9999]
slope of secant	12	12	12
t interval	[−4.01, −4]	[−4.001, −4]	[−4.0001, −4]
slope of secant	12	12	12

The slope of the tangent line at $t = -4$ is 12, coinciding with the graph of $y = f(t)$.

14. $y(x) = \dfrac{1}{x+2}$; $x = 2$

SOLUTION

x interval	[2, 2.01]	[2, 2.001]	[2, 2.0001]	[1.99, 2]	[1.999, 2]	[1.9999, 2]
slope of secant	−.0623	−.0625	−.0625	−.0627	−.0625	−.0625

The slope of the tangent line at $x = 2$ is approximately −0.06.

15. $y(t) = \sqrt{3t+1}$; $t = 1$

SOLUTION

t interval	[1, 1.01]	[1, 1.001]	[1, 1.0001]	[0.99, 1]	[0.999, 1]	[0.9999, 1]
slope of secant	.7486	.7499	.7500	.7514	.7501	.7500

The slope of the tangent line at $t = 1$ is approximately 0.75.

16. $f(x) = \sin x$; $x = \dfrac{\pi}{6}$

SOLUTION

x interval	$\left[\frac{\pi}{6} - 0.01, \frac{\pi}{6}\right]$	$\left[\frac{\pi}{6} - 0.001, \frac{\pi}{6}\right]$	$\left[\frac{\pi}{6} - 0.0001, \frac{\pi}{6}\right]$	$\left[\frac{\pi}{6}, \frac{\pi}{6} + 0.01\right]$	$\left[\frac{\pi}{6}, \frac{\pi}{6} + 0.001\right]$	$\left[\frac{\pi}{6}, \frac{\pi}{6} + 0.0001\right]$
average rate of change	0.8685	0.8663	0.8660	0.8635	0.8658	0.8660

The rate of change at $x = \frac{\pi}{6}$ is approximately 0.866.

17. $f(x) = \tan x$; $x = \dfrac{\pi}{4}$

SOLUTION

x interval	$[\frac{\pi}{4} - 0.01, \frac{\pi}{4}]$	$[\frac{\pi}{4} - 0.001, \frac{\pi}{4}]$	$[\frac{\pi}{4} - 0.0001, \frac{\pi}{4}]$	$[\frac{\pi}{4}, \frac{\pi}{4} + 0.01]$	$[\frac{\pi}{4}, \frac{\pi}{4} + 0.001]$	$[\frac{\pi}{4}, \frac{\pi}{4} + 0.0001]$
slope of secant	1.98026	1.99800	1.99980	2.02027	2.00200	2.00020

The slope of the tangent line at $x = \frac{\pi}{4}$ is approximately 2.00.

18. $f(x) = \tan x$; $x = 0$

SOLUTION

x interval	[−0.01, 0]	[−0.001, 0]	[−0.0001, 0]	[0, 0.01]	[0, 0.001]	[0, 0.0001]
slope of secant	1.00003	1.00000	1.00000	1.00003	1.00000	1.00000

The slope of the tangent line at $x = 0$ is approximately 1.00.

19. The height (in centimeters) at time t (in seconds) of a small mass oscillating at the end of a spring is $h(t) = 3\sin(2\pi t)$. Estimate its instantaneous velocity at $t = 4$.

SOLUTION To estimate the instantaneous velocity at $t = 4$, we examine the following table.

time interval	[4, 4.01]	[4, 4.001]	[4, 4.0001]	[3.99, 4]	[3.999, 4]	[3.9999, 4]
average velocity	18.8732	18.8494	18.8496	18.8732	18.8494	18.8496

The instantaneous velocity at $t = 4$ is approximately 18.85 cm/s.

20. The height (in centimeters) at time t (in seconds) of a small mass oscillating at the end of a spring is $h(t) = 8\cos(12\pi t)$.

(a) Calculate the mass's average velocity over the time intervals [0, 0.1] and [3, 3.5].

(b) Estimate its instantaneous velocity at $t = 3$.

SOLUTION

(a) The average velocity over the time interval $[t_1, t_2]$ is given by $\dfrac{\Delta h}{\Delta t} = \dfrac{h(t_2) - h(t_1)}{t_2 - t_1}$.

time interval	[0, 0.1]	[3, 3.5]
average velocity	−144.7214 cm/s	0 cm/s

(b) To estimate the instantaneous velocity at $t = 3$, we examine the following table.

time interval	[3, 3.001]	[3, 3.0001]	[3, 3.00001]	[2.999, 3]	[2.9999, 3]	[2.99999, 3]
average velocity	−5.6842	−0.5685	−0.05685	5.6842	0.5685	0.05685

The instantaneous velocity at $t = 3$ seconds is approximately 0 cm/s.

21. Consider the function $f(x) = \sqrt{x}$.

(a) Compute the slope of the secant lines from (0, 0) to $(x, f(x))$ for $x = 1, 0.1, 0.01, 0.001, 0.0001$.

(b) Discuss what the secant-line slopes in (a) suggest happens to the tangent line at 0.

(c) GU Plot the graph of f near $x = 0$ and verify your observation from (b).

SOLUTION

(a) The slope of the secant line from (0, 0) to $(x, f(x))$ for the function $f(x) = \sqrt{x}$ is

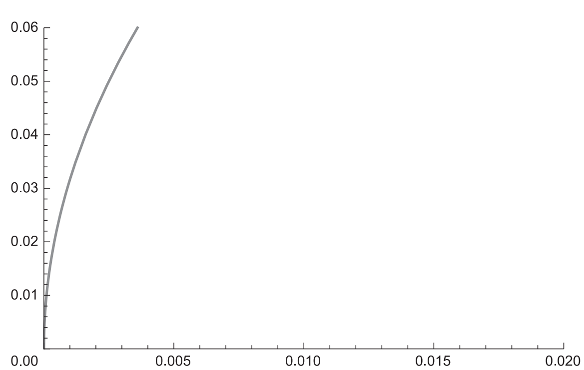
$$\frac{f(x) - f(0)}{x - 0} = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}.$$

Thus, for $x = 1, 0.1, 0.01, 0.001, 0.0001$, the slope of the secant line is

x	1	0.1	0.01	0.001	0.0001
slope of secant	1	3.16228	10	31.62278	100

(b) The secant line slopes from part (a) suggest that the slope of the tangent line at $x = 0$ grows without bound; that is, the tangent line at $x = 0$ is a vertical line.

(c) The graph of f shown below confirms that at $x = 0$ the tangent line is vertical line.



22. Consider the function $f(x) = (x-1)^{1/3}$.

(a) Compute the slope of the secant lines between 1 and x for $x = 0.9, 0.99, 0.9999$ and for $x = 1.1, 1.01, 1.0001$.

(b) Discuss what the secant-line slopes in (a) suggest happens to the tangent line at 1.

(c) GU Plot the graph of f near $x = 1$ and verify your observation from (b).

SOLUTION

(a) The slope of the secant line from 1 to x for the function $f(x) = (x-1)^{1/3}$ is

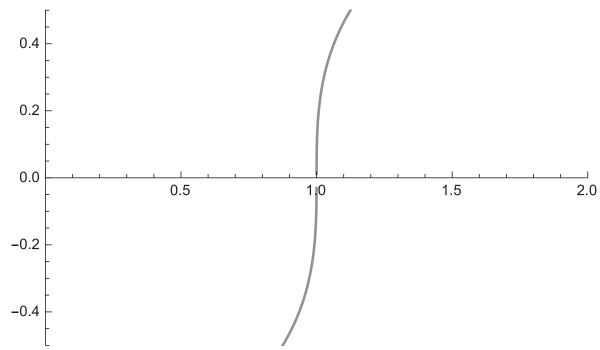
$$\frac{f(x) - f(1)}{x - 1} = \frac{(x-1)^{1/3}}{x-1} = \frac{1}{(x-1)^{2/3}}$$

Thus, for $x = 0.9, 0.99, 0.9999$ and for $x = 1.1, 1.01, 1.0001$, the slope of the secant line is

x	0.9	0.99	0.9999	1.1	1.01	1.0001
slope of secant	4.64159	21.54435	464.15888	4.64159	21.54435	464.15888

(b) The secant line slopes from part (a) suggest that the slope of the tangent line at $x = 1$ grows without bound; that is, the tangent line at $x = 0$ is a vertical line.

(c) The graph of f shown below confirms that at $x = 1$ the tangent line is vertical line.



23. If an object in linear motion (but with changing velocity) covers Δs meters in Δt seconds, then its average velocity is $v_0 = \Delta s/\Delta t$ m/s. Show that it would cover the same distance if it traveled at constant velocity v_0 over the same time interval. This justifies our calling $\Delta s/\Delta t$ the *average velocity*.

SOLUTION At constant velocity, the distance traveled is equal to velocity times time, so an object moving at constant velocity v_0 for Δt seconds travels $v_0 \delta t$ meters. Since $v_0 = \Delta s/\Delta t$, we find

$$\text{distance traveled} = v_0 \delta t = \left(\frac{\Delta s}{\Delta t}\right)\Delta t = \Delta s$$

So the object covers the same distance Δs by traveling at constant velocity v_0.

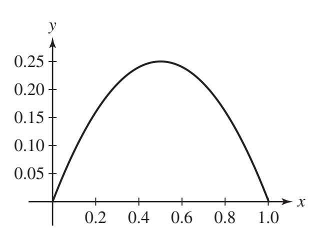
24. Sketch the graph of $f(x) = x(1-x)$ over $[0, 1]$. Refer to the graph and, without making any computations, find:

(a) The slope of the secant line over $[0, 1]$

(b) The slope of the tangent line at $x = \frac{1}{2}$

(c) The values of x at which the slope of the tangent line is positive

SOLUTION



(a) $f(0) = f(1)$, so there is no change between $x = 0$ and $x = 1$. The slope of the secant line is zero.

(b) The tangent line to the graph of $f(x)$ is horizontal at $x = \frac{1}{2}$ and therefore its slope is 0.

(c) The slope of the tangent line is positive at all points where the graph is rising. This is so for all x between $x = 0$ and $x = 0.5$.

25. Which graph in Figure 5 has the following property: For all x, the slope of the secant line over $[0, x]$ is greater than the slope of the tangent line at x. Explain.

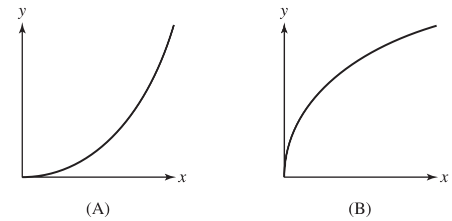


FIGURE 5

SOLUTION The graph in (B) bends downward, so the slope of the secant line through $(0, 0)$ and $(x, f(x))$ is larger than the slope of the tangent line at $(x, f(x))$. On the other hand, the graph in (A) bends upward, so the slope of the tangent line at $(x, f(x))$ is larger than the slope of the secant line through $(0, 0)$ and $(x, f(x))$. Thus, the graph in (B) has the desired property.

26. The height of a projectile fired in the air vertically with initial velocity 25 m/s is

$$h(t) = 25t - 4.9t^2 \text{ m}$$

(a) Compute $h(1)$. Show that $h(t) - h(1)$ can be factored with $(t - 1)$ as a factor.

(b) Using part (a), show that the average velocity over the interval $[1, t]$ is $20.1 - 4.9t$.

(c) Use this formula to estimate the instantaneous velocity at time $t = 1$.

SOLUTION

(a) With $h(t) = 25t - 4.9t^2$, we have $h(1) = 20.1$ m, so

$$h(t) - h(1) = -4.9t^2 + 25t - 20.1 = (t - 1)(20.1 - 4.9t).$$

(b) The average velocity over the interval $[1, t]$ is

$$\frac{h(t) - h(1)}{t - 1} = \frac{(t - 1)(20.1 - 4.9t)}{t - 1} = 20.1 - 4.9t$$

(c)

t	1.1	1.01	1.001	1.0001
average velocity over $[1, t]$	14.71	15.151	15.1951	15.19951

The instantaneous velocity is approximately 15.2 m/s. Plugging $t = 1$ second into the formula in (b) yields $20.1 - 4.9(1) = 15.2$ m/s exactly.

27. Let $Q(t) = t^2$. Find a formula for the slope of the secant line over the interval $[1, t]$ and use it to estimate the slope of the tangent line at $t = 1$. Repeat for the interval $[2, t]$ and for the slope of the tangent line at $t = 2$.

SOLUTION Let $Q(t) = t^2$. The slope of the secant line over the interval $[1, t]$ is

$$\frac{Q(t) - Q(1)}{t - 1} = \frac{t^2 - 1}{t - 1} = \frac{(t - 1)(t + 1)}{t - 1} = t + 1$$

provided $t \neq 1$. To estimate the slope of the tangent line at $t = 1$, examine the values in the table below.

t	0.99	0.999	0.9999	1.01	1.001	1.0001
slope of secant	1.99	1.999	1.9999	2.01	2.001	2.0001

The slope of the tangent line at $t = 1$ is approximately 2.0.

The slope of the secant line over the interval $[2, t]$ is

$$\frac{Q(t) - Q(2)}{t - 2} = \frac{t^2 - 4}{t - 2} = \frac{(t - 2)(t + 2)}{t - 2} = t + 2,$$

provided $t \neq 2$. To estimate the slope of the tangent line at $t = 2$, examine the values in the table below.

t	1.99	1.999	1.9999	2.01	2.001	2.0001
slope of secant	3.99	3.999	3.9999	4.01	4.001	4.0001

The slope of the tangent line at $t = 2$ is approximately 4.0.

28. For $f(x) = x^3$, show that the slope of the secant line over $[1, x]$ is $x^2 + x + 1$, and use this to estimate the slope of the tangent line at $x = 1$.

SOLUTION Let $f(x) = x^3$. The slope of the secant line over the interval $[1, x]$ is

$$\frac{f(x) - f(1)}{x - 1} = \frac{x^3 - 1}{x - 1} = \frac{(x - 1)(x^2 + x + 1)}{x - 1} = x^2 + x + 1$$

provided $x \neq 1$. To estimate the slope of the tangent line at $x = 1$, examine the values in the table below.

x	0.99	0.999	0.9999	1.01	1.001	1.0001
slope of secant	2.9701	2.997001	2.999700	3.0301	3.003001	3.000300

The slope of the tangent line at $x = 1$ is approximately 3.0.

29. For $f(x) = x^3$, show that the slope of the secant line over $[-3, x]$ is $x^2 - 3x + 9$, and use this to estimate the slope of the tangent line at $x = -3$.

SOLUTION Let $f(x) = x^3$. The slope of the secant line over the interval $[-3, x]$ is

$$\frac{f(x) - f(-3)}{x - (-3)} = \frac{x^3 + 27}{x + 3} = \frac{(x + 3)(x^2 - 3x + 9)}{x + 3} = x^2 - 3x + 9$$

provided $x \neq -3$. To estimate the slope of the tangent line at $x = -3$, examine the values in the table below.

x	−3.01	−3.001	−3.0001	−2.99	−2.999	−2.9999
slope of secant	27.0901	27.009001	27.000900	26.9101	26.991001	26.999100

The slope of the tangent line at $x = -3$ is approximately 27.0.

Further Insights and Challenges

The next two exercises involve limit estimates related to the definite integral, an important topic introduced in Chapter 5.

30. (a) Figure 6(A) shows two rectangles whose combined area is an overestimate of the area A under the graph of $y = x^2$ from $x = 0$ to $x = 1$. Compute the combined area of the rectangles.

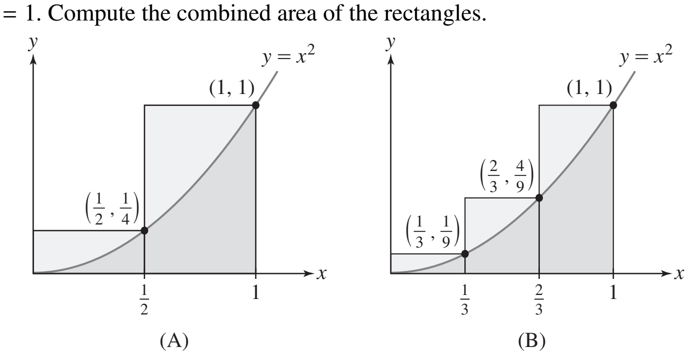


FIGURE 6

(b) We can improve the estimate by using three rectangles obtained by dividing $[0, 1]$ into thirds, as shown in Figure 6(B). Compute the combined areas of the three rectangles.

(c) Now divide $[0, 1]$ into subintervals of width $1/5$, and, on a graph of f, sketch the corresponding five rectangles obtained similar to those in (a) and (b). Compute the combined area of the five rectangles to estimate the area A.

(d) Improve your area estimate by dividing $[0, 1]$ into 10 subintervals of width $1/10$ and computing the combined area of the 10 resulting rectangles.

By dividing $[0, 1]$ into more and more subintervals, you can improve your estimate. You can use technology to carry out these computations for large numbers of rectangles. The exact value of the area is the limit of the estimates as the number of subintervals gets larger and larger.

Alternatively, for this example there is a formula (that we show how to derive in Section 5.1) that gives the total area $A(n)$ of the rectangles formed when $[0, 1]$ is divided into n subintervals of equal width:

$$A(n) = \frac{(n + 1)(2n + 1)}{6n^2}$$

(e) Compute $A(n)$ for $n = 2, 3, 5, 10$ to verify your results from (a)–(d).

(f) Compute $A(n)$ for $n = 100$, 1000, and 10,000. Use your results to conjecture what the area A equals.

SOLUTION

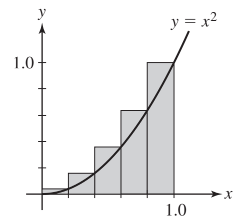
(a) The rectangle on the left in Figure 6(A) has width $\frac{1}{2}$ and height $\frac{1}{4}$, while the rectangle on the right has width $\frac{1}{2}$ and height 1. The combined area of the two rectangles is then

$$\frac{1}{2}\cdot\frac{1}{4}+\frac{1}{2}\cdot 1=\frac{5}{8}.$$

(b) The first rectangle in Figure 6(B) has width $\frac{1}{3}$ and height $\frac{1}{9}$, while the second rectangle has width $\frac{1}{3}$ and height $\frac{4}{9}$, and the third rectangle has width $\frac{1}{3}$ and height 1. The combined area of the three rectangles is then

$$\frac{1}{3}\cdot\frac{1}{9}+\frac{1}{3}\cdot\frac{4}{9}+\frac{1}{3}\cdot 1=\frac{14}{27}$$

(c) The figure below displays the graph of f together with the five rectangles obtained by dividing $[0,1]$ into five subintervals each of width $1/5$.



Each rectangle has width $1/5$, and, from left to right, the heights of the rectangles are

$$\left(\frac{1}{5}\right)^2,\left(\frac{2}{5}\right)^2,\left(\frac{3}{5}\right)^2,\left(\frac{4}{5}\right)^2,\text{ and } 1^2$$

respectively. The combined area of the five rectangles is then

$$\frac{1}{5}\left(\frac{1}{5}\right)^2+\frac{1}{5}\left(\frac{2}{5}\right)^2+\frac{1}{5}\left(\frac{3}{5}\right)^2+\frac{1}{5}\left(\frac{4}{5}\right)^2+\frac{1}{5}\cdot 1^2=\frac{55}{125}=\frac{11}{25}$$

(d) Each rectangle has width $1/10$, and, from left to right, the heights of the rectangles are

$$\left(\frac{1}{10}\right)^2,\left(\frac{1}{5}\right)^2,\left(\frac{3}{10}\right)^2,\left(\frac{2}{5}\right)^2,\left(\frac{1}{2}\right)^2,\left(\frac{3}{5}\right)^2,\left(\frac{7}{10}\right)^2,\left(\frac{4}{5}\right)^2,\left(\frac{9}{10}\right)^2,\text{ and } 1^2$$

respectively. The combined area of the five rectangles is then

$$\frac{1}{10}\left(\frac{1}{10}\right)^2+\frac{1}{10}\left(\frac{1}{5}\right)^2+\frac{1}{10}\left(\frac{3}{10}\right)^2+\frac{1}{10}\left(\frac{2}{5}\right)^2+\frac{1}{10}\left(\frac{1}{2}\right)^2+\frac{1}{10}\left(\frac{3}{5}\right)^2+\frac{1}{10}\left(\frac{7}{10}\right)^2+\frac{1}{10}\left(\frac{4}{5}\right)^2+\frac{1}{10}\left(\frac{9}{10}\right)^2+\frac{1}{10}\cdot 1^2=\frac{77}{200}$$

(e) Let

$$A(n)=\frac{(n+1)(2n+1)}{6n^2}$$

Then,

$$A(2)=\frac{3(5)}{6(4)}=\frac{5}{8}$$
$$A(3)=\frac{4(7)}{6(9)}=\frac{14}{27}$$
$$A(5)=\frac{6(11)}{6(25)}=\frac{11}{25}\text{ and}$$
$$A(10)=\frac{11(21)}{6(100)}=\frac{77}{200}$$

confirming the results from (a)–(d).

(f) We find

$$A(100)=\frac{101(201)}{6(100)^2}=0.33835$$
$$A(1000)=\frac{1001(2001)}{6(1000)^2}=0.3338335\text{ and}$$
$$A(10000)=\frac{10001(20001)}{6(10000)^2}=0.333383335$$

Based on these results, we conjecture that the area A equals $\frac{1}{3}$.

31. Let A represent the area under the graph of $y = x^3$ between $x = 0$ amd $x = 1$. In this problem, we will follow the process in Exercise 30 to approximate A.

(a) As in (a)–(d) in Exercise 30, separately divide $[0, 1]$ into 2, 3, 5, and 10 equal-width subintervals, and in each case compute an overestimate of A using rectangles on each subinterval whose height is the value of x^3 at the right end of the subinterval.

In this case, it can be shown that if we use n equal-width subintervals, then the total area $A(n)$ of the n rectangles is

$$A(n) = \frac{(n+1)^2}{4n^2}$$

(b) Compute $A(n)$ for $n = 2, 3, 5, 10$ to verify your results from (a).

(c) Compute $A(n)$ for $n = 100$, 1000, and 10,000. Use your results to conjecture what the area A equals.

SOLUTION

(a)

- Dividing $[0, 1]$ into 2 equal-width subintervals produces two rectangles with width $1/2$ and heights $1/8$ and 1. The combined area of the two rectangles is then

$$\frac{1}{2} \cdot \frac{1}{8} + \frac{1}{2} \cdot 1 = \frac{9}{16}$$

- Dividing $[0, 1]$ into 3 equal-width subintervals produces three rectangles with width $1/3$ and heights $1/27$, $8/27$, and 1. The combined area of the three rectangles is then

$$\frac{1}{3} \cdot \frac{1}{27} + \frac{1}{3} \cdot \frac{8}{27} + \frac{1}{3} \cdot 1 = \frac{4}{9}$$

- Dividing $[0, 1]$ into 5 equal-width subintervals produces five rectangles with width $1/5$ and heights

$$\left(\frac{1}{5}\right)^3, \left(\frac{2}{5}\right)^3, \left(\frac{3}{5}\right)^3, \left(\frac{4}{5}\right)^3, \text{ and } 1^3$$

respectively. The combined area of the five rectangles is then

$$\frac{1}{5}\left(\frac{1}{5}\right)^3 + \frac{1}{5}\left(\frac{2}{5}\right)^3 + \frac{1}{5}\left(\frac{3}{5}\right)^3 + \frac{1}{5}\left(\frac{4}{5}\right)^3 + \frac{1}{5} \cdot 1^3 = \frac{9}{25}$$

- Dividing $[0, 1]$ into 10 equal-width subintervals produces 10 rectangles with width $1/10$ and heights

$$\left(\frac{1}{10}\right)^3, \left(\frac{1}{5}\right)^3, \left(\frac{3}{10}\right)^3, \left(\frac{2}{5}\right)^3, \left(\frac{1}{2}\right)^3, \left(\frac{3}{5}\right)^3, \left(\frac{7}{10}\right)^3, \left(\frac{4}{5}\right)^3, \left(\frac{9}{10}\right)^3, \text{ and } 1^3$$

The combined area of the five rectangles is then

$$\frac{1}{10}\left(\frac{1}{10}\right)^3 + \frac{1}{10}\left(\frac{1}{5}\right)^3 + \frac{1}{10}\left(\frac{3}{10}\right)^3 + \frac{1}{10}\left(\frac{2}{5}\right)^3 + \frac{1}{10}\left(\frac{1}{2}\right)^3 + \frac{1}{10}\left(\frac{3}{5}\right)^3 + \frac{1}{10}\left(\frac{7}{10}\right)^3 + \frac{1}{10}\left(\frac{4}{5}\right)^3 + \frac{1}{10}\left(\frac{9}{10}\right)^3 + \frac{1}{10} \cdot 1^3 = \frac{121}{400}$$

(b) Let

$$A(n) = \frac{(n+1)^2}{4n^2}$$

Then,

$$A(2) = \frac{9}{4(4)} = \frac{9}{16}$$

$$A(3) = \frac{16}{4(9)} = \frac{4}{9}$$

$$A(5) = \frac{36}{4(25)} = \frac{9}{25}, \text{ and}$$

$$A(10) = \frac{121)}{4(100)} = \frac{121}{400}$$

confirming the results from (a).

(c) We find

$$A(100) = \frac{101^2}{4(100)^2} = 0.255025$$

$$A(1000) = \frac{1001^2}{4(1000)^2} = 0.25050025, \text{ and}$$

$$A(10000) = \frac{10001^2}{4(10000)^2} = 0.2500500025$$

Based on these results, we conjecture that the area A equals $\frac{1}{4}$.

2.2 Investigating Limits

Preliminary Questions

1. What is the limit of $f(x) = 1$ as $x \to \pi$?

SOLUTION $\lim_{x\to\pi} 1 = 1$.

2. What is the limit of $g(t) = t$ as $t \to \pi$?

SOLUTION $\lim_{t\to\pi} t = \pi$.

3. Is $\lim_{x\to 10} 20$ equal to 10 or 20?

SOLUTION $\lim_{x\to 10} 20 = 20$.

4. Can $f(x)$ approach a limit as $x \to c$ if $f(c)$ is undefined? If so, give an example.

SOLUTION Yes. The limit of a function f as $x \to c$ does not depend on what happens *at* $x = c$, only on the behavior of f as $x \to c$. As an example, consider the function

$$f(x) = \frac{x^2 - 1}{x - 1}$$

The function is clearly not defined at $x = 1$ but

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2 - 1}{x - 1} = \lim_{x\to 1}(x + 1) = 2$$

5. What does the following table suggest about $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$?

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	7	25	4317	3.00011	3.0047	3.0126

SOLUTION The values in the table suggest that $\lim_{x\to 1^-} f(x) = \infty$ and $\lim_{x\to 1^+} f(x) = 3$.

6. Can you tell whether $\lim_{x\to 5} f(x)$ exists from a plot of f for $x > 5$? Explain.

SOLUTION No. By examining values of $f(x)$ for x close to but greater than 5, we can determine whether the one-sided limit $\lim_{x\to 5^+} f(x)$ exists. To determine whether $\lim_{x\to 5} f(x)$ exists, we must examine value of $f(x)$ on both sides of $x = 5$.

7. If you know in advance that $\lim_{x\to 5} f(x)$ exists, can you determine its value from a plot of f for all $x > 5$?

SOLUTION Yes. If $\lim_{x\to 5} f(x)$ exists, then both one-sided limits must exist and be equal.

Exercises

In Exercises 1–5, fill in the table and guess the value of the limit.

1. $\lim_{x\to 1} f(x)$, where $f(x) = \dfrac{x^3 - 1}{x^2 - 1}$

x	$f(x)$	x	$f(x)$
1.002		0.998	
1.001		0.999	
1.0005		0.9995	
1.00001		0.99999	

SOLUTION

x	0.998	0.999	0.9995	0.99999	1.00001	1.0005	1.001	1.002
$f(x)$	1.498501	1.499250	1.499625	1.499993	1.500008	1.500375	1.500750	1.501500

The limit as $x \to 1$ is $\frac{3}{2}$.

2. $\lim_{t\to 0} h(t)$, where $h(t) = \dfrac{\cos t - 1}{t^2}$. Note that h is even; that is, $h(t) = h(-t)$.

t	± 0.002	± 0.0001	± 0.00005	± 0.00001
$h(t)$				

SOLUTION

t	± 0.002	± 0.0001
$h(t)$	-0.499999833333	-0.499999999583
t	± 0.00005	± 0.00001
$h(t)$	-0.499999999896	-0.500000000000

The limit as $t \to 0$ is $-\frac{1}{2}$.

3. $\lim_{y\to 2} f(y)$, where $f(y) = \dfrac{y^2 - y - 2}{y^2 + y - 6}$

y	$f(y)$	y	$f(y)$
2.002		1.998	
2.001		1.999	
2.0001		1.9999	

SOLUTION

y	1.998	1.999	1.9999	2.0001	2.001	2.002
$f(y)$	0.59984	0.59992	0.599992	0.600008	0.60008	0.60016

The limit as $y \to 2$ is $\frac{3}{5}$.

4. $\lim_{\theta\to 0} f(\theta)$, where $f(\theta) = \dfrac{\sin\theta - \theta}{\theta^3}$.

θ	± 0.002	± 0.0001	± 0.00005	± 0.00001
$f(\theta)$				

SOLUTION

θ	± 0.002	± 0.0001
$f(\theta)$	-0.1666666333	-0.1666666666
θ	± 0.00005	± 0.00001
$f(\theta)$	-0.1666666666	-0.1666666667

The limit as $\theta \to 0$ is $-\frac{1}{6}$.

5. $\lim_{t\to 0} f(t)$, where $f(t) = \dfrac{1 - \cos 2t}{t}$

t	$f(t)$	t	$f(t)$
0.002		-0.002	
0.001		-0.001	
0.0005		-0.0005	
0.00001		-0.00001	

SOLUTION

t	$f(t)$	t	$f(t)$
0.002	0.004	−0.002	−0.004
0.001	0.002	−0.001	−0.002
0.0005	0.001	−0.0005	−0.001
0.00001	0.00002	−0.00001	−0.00002

The limit as $t \to 0$ is 0.

6. Numerically investigate $\lim_{x\to 0} \frac{\sin x}{x}$, computing the values of $\sin x$ with x in degrees. Make an estimate of the limit accurate to 5 decimal places.

SOLUTION Let $f(x) = \frac{\sin x}{x}$ with x measured in degrees. Then

x	$f(x)$	x	$f(x)$
1	0.0174524	−1	0.0174524
0.1	0.0174533	−0.1	0.0174533
0.01	0.0174533	−0.01	0.0174533
0.001	0.0174533	−0.001	0.0174533

Based on the values in this table, $\lim_{x\to 0} \frac{\sin x}{x} \approx 0.01745$, accurate to five decimal places.

7. Determine $\lim_{x\to 0.5} f(x)$ for f as in Figure 10.

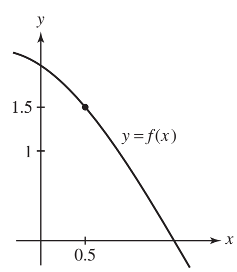


FIGURE 10

SOLUTION The graph suggests that $f(x) \to 1.5$ as $x \to 0.5$.

8. Determine $\lim_{x\to 0.5} g(x)$ for g as in Figure 11.

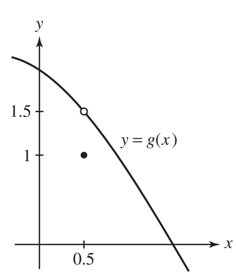


FIGURE 11

SOLUTION The graph suggests that $g(x) \to 1.5$ as $x \to .5$. The value $g(1.5)$, which happens to be 1, does not affect the limit.

In Exercises 9–10, evaluate the limit.

9. $\lim_{x\to 21} x$

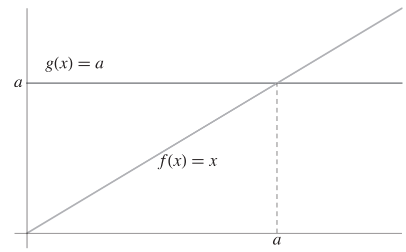
SOLUTION As $x \to 21$, $f(x) = x \to 21$. You can see this, for example, on the graph of $f(x) = x$.

10. $\lim_{x\to 4.2} \sqrt{3}$

SOLUTION The graph of $f(x) = \sqrt{3}$ is a horizontal line. $f(x) = \sqrt{3}$ for all values of x, so the limit is also equal to $\sqrt{3}$.

11. Show, via illustration, that the limits $\lim_{x\to a} x$ and $\lim_{x\to a} a$ are equal but the functions in each limit are different.

SOLUTION



The figure above displays the graphs of $f(x) = x$ and $g(x) = a$. Clearly, the two functions are different. It is also clear that as x approaches a, both graphs approach the point (a, a); that is,

$$\lim_{x\to a} x = \lim_{x\to a} a = a$$

12. Give examples of functions f and g such that $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x)$, but $f(x) \neq g(x)$ for all x, including 0.

SOLUTION There are many possible pairs of functions that satisfy these conditions; here is one possibility. Let $f(x) = x^2$ and

$$g(x) = \begin{cases} -x^2 & \text{when} \quad x \neq 0 \\ -1 & \text{when} \quad x = 0 \end{cases}$$

There is no value of x for which $f(x) = g(x)$, but

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$$

In Exercises 13–20, verify each limit using the limit definition. For example, in Exercise 13, show that $|3x - 12|$ *can be made as small as desired by taking* x *close to* 4.

13. $\lim_{x\to 4} 3x = 12$

SOLUTION $|3x - 12| = 3|x - 4|$. $|3x - 12|$ can be made arbitrarily small by making x close enough to 4, thus making $|x - 4|$ small.

14. $\lim_{x\to 5} 3 = 3$

SOLUTION $|f(x) - 3| = |3 - 3| = 0$ for all values of x so $f(x) - 3$ is already smaller than any positive number as $x \to 5$.

15. $\lim_{x\to 3}(5x + 2) = 17$

SOLUTION $|(5x + 2) - 17| = |5x - 15| = 5|x - 3|$. Therefore, if you make $|x - 3|$ small enough, you can make $|(5x + 2) - 17|$ as small as desired.

16. $\lim_{x\to 2}(7x - 4) = 10$

SOLUTION As $x \to 2$, note that $|(7x - 4) - 10| = |7x - 14| = 7\,|x - 2|$. If you make $|x - 2|$ small enough, you can make $|(7x - 4) - 10|$ as small as desired.

17. $\lim_{x\to 0} x^2 = 0$

SOLUTION As $x \to 0$, we have $|x^2 - 0| = |x + 0||x - 0|$. To simplify things, suppose that $|x| < 1$, so that $|x + 0||x - 0| = |x||x| < |x|$. By making $|x|$ sufficiently small, so that $|x + 0||x - 0| = x^2$ is even smaller, you can make $|x^2 - 0|$ as small as desired.

18. $\lim_{x\to 0}(3x^2 - 9) = -9$

SOLUTION $|3x^2 - 9 - (-9)| = |3x^2| = 3|x^2|$. If you make $|x| < 1$, $|x^2| < |x|$, so that making $|x - 0|$ small enough can make $|3x^2 - 9 - (-9)|$ as small as desired.

19. $\lim_{x\to 0}(4x^2 + 2x + 5) = 5$

SOLUTION As $x \to 0$, we have $|4x^2 + 2x + 5 - 5| = |4x^2 + 2x| = |x||4x + 2|$. If $|x| < 1$, $|4x + 2|$ can be no bigger than 6, so $|x||4x + 2| < 6|x|$. Therefore, by making $|x - 0| = |x|$ sufficiently small, you can make $|4x^2 + 2x + 5 - 5| = |x||4x + 2|$ as small as desired.

20. $\lim_{x\to 0}(x^3 + 12) = 12$

SOLUTION $|(x^3 + 12) - 12| = |x^3|$. If we make $|x| < 1$, then $|x^3| < |x|$. Therefore, by making $|x - 0| = |x|$ sufficiently small, we can make $|(x^3 + 12) - 12|$ as small as desired.

In Exercises 21–42, estimate the limit numerically or state that the limit does not exist. If infinite, state whether the one-sided limits are ∞ or $-\infty$.

21. $\lim_{x\to 1}\dfrac{\sqrt{x} - 1}{x - 1}$

SOLUTION

x	0.9995	0.99999	1.00001	1.0005
$f(x)$	0.500063	0.500001	0.49999	0.499938

The limit as $x \to 1$ is $\frac{1}{2}$.

22. $\lim_{x\to -4}\dfrac{2x^2 - 32}{x + 4}$

SOLUTION

x	−4.0005	−4.00001	−3.99999	−3.9995
$f(x)$	−16.001	−16.00002	−15.99998	−15.999

The limit as $x \to -4$ is -16.

23. $\lim_{x\to 2}\dfrac{x^2 + x - 6}{x^2 - x - 2}$

SOLUTION

x	1.999	1.99999	2.00001	2.001
$f(x)$	1.666889	1.666669	1.666664	1.666445

The limit as $x \to 2$ is $\frac{5}{3}$.

24. $\lim_{x\to 3}\dfrac{x^3 - 2x^2 - 9}{x^2 - 2x - 3}$

SOLUTION

x	2.99	2.995	3.005	3.01
$f(x)$	3.741880	3.745939	3.754064	3.758130

The limit as $x \to 3$ is 3.75.

25. $\lim_{x\to 0}\dfrac{\sin 2x}{x}$

SOLUTION

x	−0.01	−0.005	0.005	0.01
$f(x)$	1.999867	1.999967	1.999967	1.999867

The limit as $x \to 0$ is 2.

26. $\lim_{x\to 0}\dfrac{\sin 5x}{x}$

SOLUTION

x	−0.01	−0.005	0.005	0.01
$f(x)$	4.997917	4.999479	4.999479	4.997917

The limit as $x \to 0$ is 5.

27. $\lim_{x\to 0} \frac{\sin 3x}{3x}$

SOLUTION

x	-0.01	-0.001	0.001	0.01
$f(x)$	0.999850	0.999999	0.999999	0.999850

The limit as $x \to 0$ is 1.

28. $\lim_{x\to 0} \frac{\cos x}{3x}$

SOLUTION

x	-0.01	-0.001	-0.0001	0.0001	0.001	0.01
$f(x)$	-33.3317	-333.3332	-3333.3333	3333.3333	333.3332	33.3317

The limit does not exist. As $x \to 0^-$, $f(x) \to -\infty$; similarly, as $x \to 0^+$, $f(x) \to \infty$.

29. $\lim_{\theta\to 0} \frac{\cos\theta - 1}{\theta}$

SOLUTION

x	-0.05	-0.001	0.001	0.05
$f(x)$	0.0249948	0.0005	-0.0005	-0.0249948

The limit as $x \to 0$ is 0.

30. $\lim_{x\to 0} \frac{\sin x}{x^2}$

SOLUTION

x	-0.01	-0.001	-0.0001	0.0001	0.001	0.01
$f(x)$	-99.9983	-999.9998	-10000.0	10000.0	999.9998	99.9983

The limit does not exist. As $x \to 0^-$, $f(x) \to -\infty$; similarly, as $x \to 0^+$, $f(x) \to \infty$.

31. $\lim_{x\to 4} \frac{1}{(x-4)^3}$

SOLUTION

x	3.9	3.99	3.999	4.001	4.01	4.1
$f(x)$	-1000	-10^6	-10^9	10^9	10^6	1000

The limit does not exist. As $x \to 4^-$, $f(x) \to -\infty$; similarly, as $x \to 4^+$, $f(x) \to \infty$.

32. $\lim_{x\to 1^-} \frac{3-x}{x-1}$

SOLUTION

x	0.9	0.99	0.999
$f(x)$	-21	-201	-2001

The limit does not exist. As $x \to 1^-$, $f(x) \to -\infty$.

33. $\lim_{x\to -3} \frac{x+3}{x^2+x-6}$

SOLUTION

x	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9
$f(x)$	-0.196078	-0.199601	-0.199960	-0.200040	-0.200401	-0.204082

The limit as $x \to -3$ is $-\frac{1}{5}$.

34. $\lim_{x \to -2^-} \frac{x+1}{x+2}$

SOLUTION

x	−2.1	−2.01	−2.001
$f(x)$	11	101	1001

The limit does not exist. As $x \to -2^-$, $f(x) \to \infty$.

35. $\lim_{x \to 3^+} \frac{x-4}{x^2-9}$

SOLUTION

x	3.001	3.01	3.1
$f(x)$	−166.472	−16.473	−1.475

The limit does not exist. As $x \to 3^+$, $f(x) \to -\infty$.

36. $\lim_{h \to 0} \frac{3^h - 1}{h}$

SOLUTION

h	−0.05	−0.001	−0.0001	0.0001	0.001	0.05
$f(h)$	1.068984	1.098009	1.098552	1.098673	1.099216	1.129346

The limit as $x \to 0$ is approximately 1.099. (The exact answer is $\ln 3$.)

37. $\lim_{h \to 0} \sin h \cos \frac{1}{h}$

SOLUTION

h	−0.01	−0.001	−0.0001	0.0001	0.001	0.01
$f(h)$	−0.008623	−0.000562	0.000095	−0.000095	0.000562	0.008623

The limit as $x \to 0$ is 0.

38. $\lim_{h \to 0} \cos \frac{1}{h}$

SOLUTION

h	±0.1	±0.01	±0.001	±0.0001
$f(h)$	−0.839072	0.862319	0.562379	−0.952155

The limit does not exist since $\cos(1/h)$ oscillates infinitely often as $h \to 0$.

39. $\lim_{x \to 0} |x|^x$

SOLUTION

x	−0.05	−0.001	−0.00001	0.00001	0.001	0.05
$f(x)$	1.161586	1.006932	1.000115	0.999885	0.993116	0.860892

The limit as $x \to 0$ is 1.

40. $\lim_{r \to 0} (1+2r)^{1/r}$

SOLUTION

x	−0.001	−0.0001	−0.00001	0.00001	0.0001	0.001
$f(x)$	7.403869	7.390534	7.389204	7.388908	7.387579	7.374312

The limit as $r \to 0$ is approximately 7.389. (The exact answer is e^2.)

41. $\lim_{\theta \to \pi/4} \dfrac{\tan\theta - 2\sin\theta\cos\theta}{\theta - \pi/4}$

SOLUTION

θ	$\pi/4 - 0.01$	$\pi/4 - 0.001$	$\pi/4 - 0.0001$	$\pi/4 + 0.0001$	$\pi/4 + 0.001$	$\pi/4 + 0.01$
$f(\theta)$	1.960264	1.996003	1.999600	2.000400	2.004003	2.040269

The limit as $\theta \to \pi/4$ is approximately 2.000. (The exact answer is 2.)

42. $\lim_{x \to 0} \dfrac{\tan x - x}{\sin x - x}$

SOLUTION

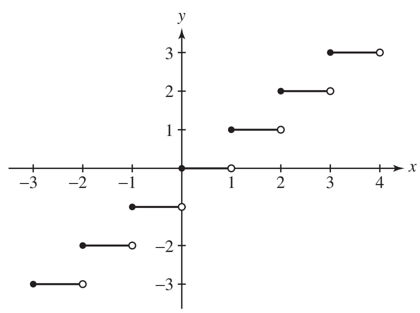
x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	-2.009037	-2.000090	-2.000001	-2.000001	-2.000090	-2.009037

The limit as $x \to 0$ is approximately -2.000. (The exact answer is -2.)

43. The **greatest integer function**, also known as the **floor function**, is defined by $\lfloor x \rfloor = n$, where n is the unique integer such that $n \le x < n + 1$. Sketch the graph of $y = \lfloor x \rfloor$. Calculate for c an integer:

(a) $\lim_{x \to c^-} \lfloor x \rfloor$ **(b)** $\lim_{x \to c^+} \lfloor x \rfloor$ **(c)** $\lim_{x \to 2.6} \lfloor x \rfloor$

SOLUTION The graph of $y = \lfloor x \rfloor$ is shown below.



(a) From the graph of the greatest integer function, we see that $\lim_{x \to c^-} \lfloor x \rfloor = c - 1$, where c is an integer.

(b) Again from the graph of the greatest integer function, we see that $\lim_{x \to c^+} \lfloor x \rfloor = c$, where c is an integer.

(c) Examining the graph in part (a), we see that $\lim_{x \to 2.6} \lfloor x \rfloor = 2$.

44. Determine the one-sided limits at $c = 1$, 2, and 4 of the function g shown in Figure 12, and state whether the limit exists at these points.

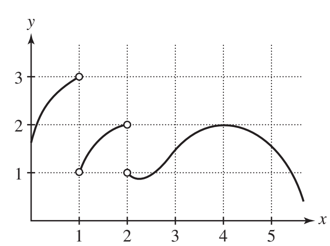


FIGURE 12

SOLUTION Based on Figure 12,

$$\lim_{x \to 1^-} g(x) = 3, \qquad \text{while} \qquad \lim_{x \to 1^+} g(x) = 1.$$

Because these two one-sided limits are not equal, $\lim_{x \to 1} g(x)$ does not exist. Next,

$$\lim_{x \to 2^-} g(x) = 2, \qquad \text{while} \qquad \lim_{x \to 2^+} g(x) = 1.$$

Because these two one-sided limits are not equal, $\lim_{x \to 2} g(x)$ does not exist. Finally,

$$\lim_{x \to 4^-} g(x) = 2, \qquad \text{while} \qquad \lim_{x \to 4^+} g(x) = 2.$$

Because these two one-sided limits are equal, $\lim_{x \to 4} g(x)$ does exist and $\lim_{x \to 4} g(x) = 2$.

In Exercises 45–52, determine the one-sided limits numerically or graphically. If infinite, state whether the one-sided limits are ∞ or −∞, and describe the corresponding vertical asymptote. In Exercise 52, $f(x) = \lfloor x \rfloor$ is the greatest integer function defined in Exercise 43.

45. $\lim_{x\to 0^{\pm}} \dfrac{\sin x}{|x|}$

SOLUTION

x	−0.2	−0.02	0.02	0.2
$f(x)$	−0.993347	−0.999933	0.999933	0.993347

The left-hand limit is $\lim_{x\to 0^-} f(x) = -1$, whereas the right-hand limit is $\lim_{x\to 0^+} f(x) = 1$.

46. $\lim_{x\to 0^{\pm}} |x|^{1/x}$

SOLUTION

x	−0.2	−0.1	0.15	0.2
$f(x)$	3125.0	10^{10}	0.000003	0.000320

The left-hand limit is $\lim_{x\to 0^-} f(x) = \infty$, whereas the right-hand limit is $\lim_{x\to 0^+} f(x) = 0$. Because $\lim_{x\to 0^-} f(x) = \infty$, the line $x = 0$ is a vertical asymptote.

47. $\lim_{x\to 0^{\pm}} \dfrac{x - \sin|x|}{x^3}$

SOLUTION

x	−0.1	−0.01	0.01	0.1
$f(x)$	199.853	19999.8	0.166666	0.166583

The left-hand limit is $\lim_{x\to 0^-} f(x) = \infty$, whereas the right-hand limit is $\lim_{x\to 0^+} f(x) = \frac{1}{6}$. Because $\lim_{x\to 0^-} f(x) = \infty$, the line $x = 0$ is a vertical asymptote.

48. $\lim_{x\to 4^{\pm}} \dfrac{x+1}{x-4}$

SOLUTION

x	3.99	3.999	4.001	4.01
$f(x)$	−499	−4999	5001	501

The left-hand limit is $\lim_{x\to 4^-} f(x) = -\infty$, whereas the right-hand limit is $\lim_{x\to 4^+} f(x) = \infty$. Because the one-sided limits are infinite, the line $x = 4$ is a vertical asymptote.

49. $\lim_{x\to -2^{\pm}} \dfrac{4x^2+7}{x^3+8}$

SOLUTION

x	−2.1	−2.01	−1.99	−1.9
$f(x)$	−19.540048	−192.041525	191.291530	18.790535

The left-hand limit is $\lim_{x\to -2^-} f(x) = -\infty$, whereas the right-hand limit is $\lim_{x\to -2^+} f(x) = \infty$. Because the one-sided limits are infinite, the line $x = -2$ is a vertical asymptote.

50. $\lim_{x\to -3^{\pm}} \dfrac{x^2}{x^2-9}$

SOLUTION

x	−3.01	−3.001	−2.999	−2.99
$f(x)$	150.750416	1500.750042	−1499.250042	−149.250417

The left-hand limit is $\lim_{x\to -3^-} f(x) = \infty$, whereas the right-hand limit is $\lim_{x\to -3^+} f(x) = -\infty$. Because the one-sided limits are infinite, the line $x = -3$ is a vertical asymptote.

51. $\lim_{x\to 1^{\pm}} \frac{x^5 + x - 2}{x^2 + x - 2}$

SOLUTION

x	0.99	0.999	1.001	1.01
$f(x)$	1.973577	1.997336	2.002669	2.026912

The left-hand limit is $\lim_{x\to 1^-} f(x) = 2$, whereas the right-hand limit is $\lim_{x\to 1^+} f(x) = 2$.

52. $\lim_{x\to 2^{\pm}} \cos\left(\frac{\pi}{2}(x - \lfloor x \rfloor)\right)$

SOLUTION

x	1.99	1.999	2.001	2.01
$f(x)$	0.015707	0.001571	0.999999	0.999877

The left-hand limit is $\lim_{x\to 2^-} f(x) = 0$, whereas the right-hand limit is $\lim_{x\to 2^+} f(x) = 1$.

53. Determine the one-sided limits at $c = 2$ and $c = 4$ of the function f in Figure 13. What are the vertical asymptotes of f?

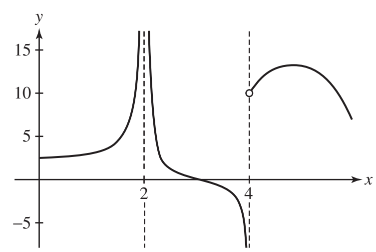


FIGURE 13

SOLUTION

- For $c = 2$, we have $\lim_{x\to 2^-} f(x) = \infty$ and $\lim_{x\to 2^+} f(x) = \infty$.
- For $c = 4$, we have $\lim_{x\to 4^-} f(x) = -\infty$ and $\lim_{x\to 4^+} f(x) = 10$.

The vertical asymptotes are the vertical lines $x = 2$ and $x = 4$.

54. Determine the infinite one- and two-sided limits in Figure 14.

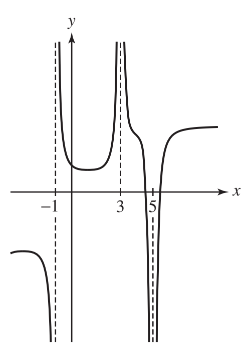


FIGURE 14

SOLUTION

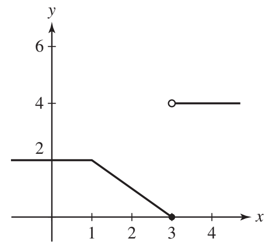
- $\lim_{x\to -1^-} f(x) = -\infty$
- $\lim_{x\to -1^+} f(x) = \infty$
- $\lim_{x\to 3} f(x) = \infty$
- $\lim_{x\to 5} f(x) = -\infty$

The vertical asymptotes are the vertical lines $x = 1$, $x = 3$, and $x = 5$.

In Exercises 55–58, sketch the graph of a function with the given limits.

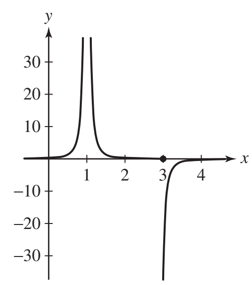
55. $\lim_{x\to 1} f(x) = 2, \quad \lim_{x\to 3^-} f(x) = 0, \quad \lim_{x\to 3^+} f(x) = 4$

SOLUTION



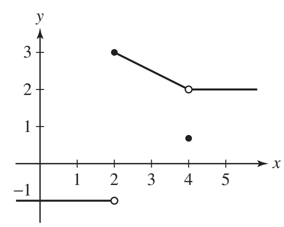
56. $\lim_{x\to 1} f(x) = \infty, \quad \lim_{x\to 3^-} f(x) = 0, \quad \lim_{x\to 3^+} f(x) = -\infty$

SOLUTION



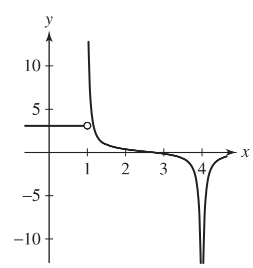
57. $\lim_{x\to 2^+} f(x) = f(2) = 3, \quad \lim_{x\to 2^-} f(x) = -1, \quad \lim_{x\to 4} f(x) = 2 \neq f(4)$

SOLUTION



58. $\lim_{x\to 1^+} f(x) = \infty, \quad \lim_{x\to 1^-} f(x) = 3, \quad \lim_{x\to 4} f(x) = -\infty$

SOLUTION



59. Determine the one-sided limits of the function f in Figure 15, at the points $c = 1, 3, 5, 6$.

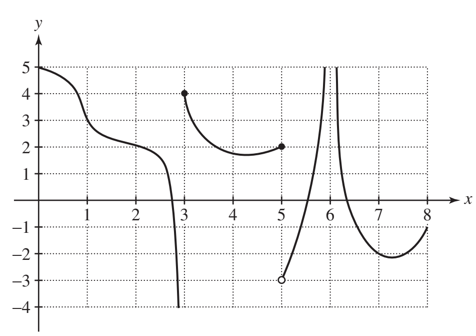


FIGURE 15 Graph of f.

SOLUTION Based on the graph of the function f in Figure 15,

$$\lim_{x\to 1^-} f(x) = 3, \qquad \lim_{x\to 1^+} f(x) = 3,$$
$$\lim_{x\to 3^-} f(x) = -\infty, \qquad \lim_{x\to 3^+} f(x) = 4,$$
$$\lim_{x\to 5^-} f(x) = 2, \qquad \lim_{x\to 5^+} f(x) = -3,$$
$$\lim_{x\to 6^-} f(x) = \infty, \text{ and } \lim_{x\to 6^+} f(x) = \infty.$$

60. Does either of the two oscillating functions in Figure 16 appear to approach a limit as $x \to 0$?

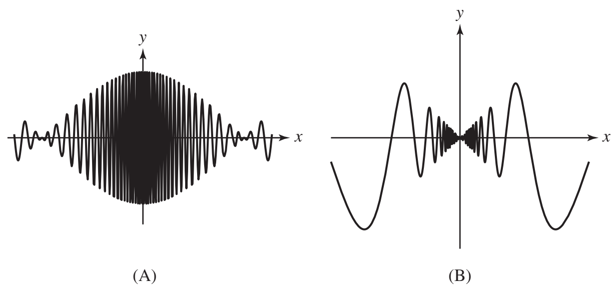


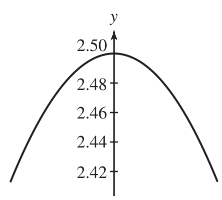
FIGURE 16

SOLUTION (A) does not appear to approach a limit as $x \to 0$; the values of the function oscillate wildly as $x \to 0$. The values of the function graphed in (B) seem to settle to 0 as $x \to 0$, so the limit seems to exist.

GU *In Exercises 61–66, plot the function and use the graph to estimate the value of the limit.*

61. $\displaystyle\lim_{\theta\to 0} \frac{\sin 5\theta}{\sin 2\theta}$

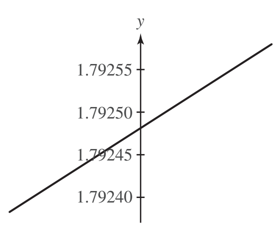
SOLUTION



The limit as $\theta \to 0$ is $\frac{5}{2}$.

62. $\displaystyle\lim_{x\to 0} \frac{12^x - 1}{4^x - 1}$

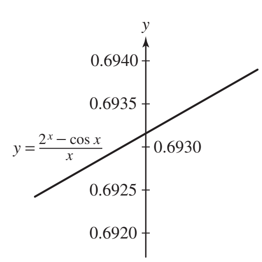
SOLUTION



The limit as $\theta \to 0$ is approximately 1.7925. (The exact value is $\frac{\ln 12}{\ln 4}$.)

63. $\displaystyle\lim_{x\to 0} \frac{2^x - \cos x}{x}$

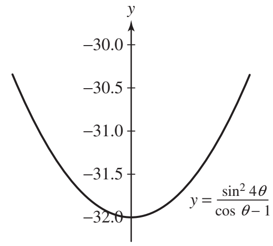
SOLUTION



The limit as $x \to 0$ is approximately 0.693. (The exact answer is $\ln 2$.)

64. $\lim_{\theta \to 0} \frac{\sin^2 4\theta}{\cos\theta - 1}$

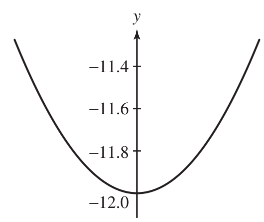
SOLUTION



The limit as $\theta \to 0$ is -32.

65. $\lim_{\theta \to 0} \frac{\cos 7\theta - \cos 5\theta}{\theta^2}$

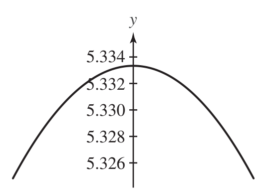
SOLUTION



The limit as $\theta \to 0$ is -12.

66. $\lim_{\theta \to 0} \frac{\sin^2 2\theta - \theta \sin 4\theta}{\theta^4}$

SOLUTION



The limit as $\theta \to 0$ is approximately 5.333. (The exact value is $\frac{16}{3}$.)

67. Let n be a positive integer. For which n are the two infinite one-sided limits $\lim_{x \to 0\pm} 1/x^n$ equal?

SOLUTION If $x > 0$, then $x^n > 0$ for any positive integer n. Moreover, as $x \to 0^+$, $x^n \to 0^+$, so

$$\lim_{x \to 0^+} \frac{1}{x^n} = \infty$$

for any positive integer n. On the other hand, if $x < 0$, then $x^n < 0$ when n is an odd positive integer and $x^n > 0$ when n is an even positive integer. Accordingly,

$$\lim_{x \to 0^-} \frac{1}{x^n} = \begin{cases} -\infty, & n \text{ is an odd positive integer} \\ \infty, & n \text{ is an even positive integer.} \end{cases}$$

Thus, the two infinite one-sided limits $\lim_{x \to 0\pm} 1/x^n$ are equal when n is an even positive integer.

68. Let $L(n) = \lim_{x \to 1} \left(\frac{n}{1 - x^n} - \frac{1}{1 - x} \right)$ for n a positive integer. Investigate $L(n)$ numerically for several values of n, and then guess the value of $L(n)$ in general.

SOLUTION

- For $n = 1$,

$$L(1) = \lim_{x \to 1} \left(\frac{1}{1 - x} - \frac{1}{1 - x} \right) = \lim_{x \to 1} 0 = 0$$

- For $n = 3$, we have

x	0.9	0.99	0.999	1.001	1.01	1.1
$\dfrac{3}{1-x^3} - \dfrac{1}{1-x}$	1.070111	1.006700	1.000667	0.999334	0.993367	0.936556

The limit as $x \to 1$ is 1.

- For $n = 6$, we have

x	0.9	0.99	0.999	1.001	1.01	1.1
$\dfrac{6}{1-x^6} - \dfrac{1}{1-x}$	2.805218	2.529312	2.502918	2.497084	2.470980	2.223557

The limit as $x \to 1$ is 2.5.

- We surmise that, in general, $\lim_{x\to 1} L(n) = \dfrac{n-1}{2}$.

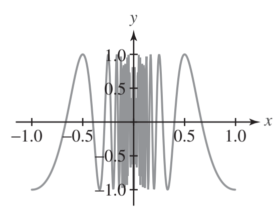
69. GU In some cases, numerical investigations can be misleading. Plot $f(x) = \cos \frac{\pi}{x}$.

(a) Does $\lim_{x\to 0} f(x)$ exist?

(b) Show, by evaluating $f(x)$ at $x = \pm\frac{1}{2}, \pm\frac{1}{4}, \pm\frac{1}{6}, \dots$, that you might be able to trick your friends into believing that the limit exists and is equal to $L = 1$.

(c) Which sequence of evaluations might trick them into believing that the limit is $L = -1$?

SOLUTION A graph of f is shown below:



(a) Based on the graph of f, it appears that the function values oscillate more and more rapidly between $+1$ and -1 as $x \to 0$. Accordingly, it appears that $\lim_{x\to 0} f(x)$ does not exist.

(b) Evaluating $f(x)$ at $x = \pm\frac{1}{2}, \pm\frac{1}{4}, \pm\frac{1}{6}, \dots$, we find

$$f\left(\pm\frac{1}{2}\right) = \cos(\pm 2\pi) = 1$$

$$f\left(\pm\frac{1}{4}\right) = \cos(\pm 4\pi) = 1$$

$$f\left(\pm\frac{1}{6}\right) = \cos(\pm 6\pi) = 1$$

and so on.

(c) To trick your friends into believing that $L = -1$, evaluate $f(x)$ at $x = \pm 1, \pm\frac{1}{3}, \pm\frac{1}{5}, \dots$.

Further Insights and Challenges

70. Light waves of frequency λ passing through a slit of width a produce a **Fraunhofer diffraction pattern** of light and dark fringes (Figure 17). The intensity as a function of the angle θ is

$$I(\theta) = I_m \left(\frac{\sin(R\sin\theta)}{R\sin\theta}\right)^2$$

where $R = \pi a/\lambda$ and I_m is a constant. Show that the intensity function is not defined at $\theta = 0$. Then choose any two values for R and check numerically that $I(\theta)$ approaches I_m as $\theta \to 0$.

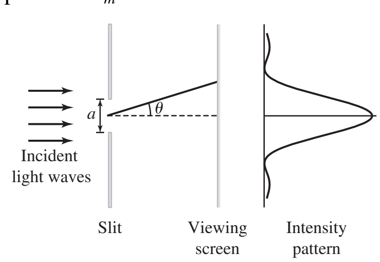


FIGURE 17 Fraunhofer diffraction pattern.

SOLUTION If you plug in $\theta = 0$, you get a division by zero in the expression

$$\frac{\sin\left(R\sin\theta\right)}{R\sin\theta}$$

thus, $I(0)$ is undefined. If $R = 2$, a table of values as $\theta \to 0$ follows:

θ	-0.01	-0.005	0.005	0.01
$I(\theta)$	$0.998667\ I_m$	$0.9999667\ I_m$	$0.9999667\ I_m$	$0.9998667\ I_m$

The limit as $\theta \to 0$ is $1 \cdot I_m = I_m$.

If $R = 3$, the table becomes

θ	-0.01	-0.005	0.005	0.01
$I(\theta)$	$0.999700\ I_m$	$0.999925\ I_m$	$0.999925\ I_m$	$0.999700\ I_m$

Again, the limit as $\theta \to 0$ is $1I_m = I_m$.

71. Investigate $\lim_{\theta\to 0} \frac{\sin n\theta}{\theta}$ numerically for several positive integer values of n. Then guess the value in general.

SOLUTION

- For $n = 3$, we have

θ	-0.1	-0.01	-0.001	0.001	0.01	0.1
$\frac{\sin n\theta}{\theta}$	2.955202	2.999550	2.999996	2.999996	2.999550	2.955202

  The limit as $\theta \to 0$ is 3.
- For $n = 5$, we have

θ	-0.1	-0.01	-0.001	0.001	0.01	0.1
$\frac{\sin n\theta}{\theta}$	4.794255	4.997917	4.999979	4.999979	4.997917	4.794255

  The limit as $\theta \to 0$ is 5.
- We surmise that, in general, $\lim_{\theta\to 0} \frac{\sin n\theta}{\theta} = n$.

72. Show numerically that $\lim_{x\to 0} \frac{b^x - 1}{x}$ is less than 2 with $b = 7$ and is greater than 2 with $b = 8$. Experiment with values of b to find an approximate value of b for which the limit is 2.

SOLUTION Based on the first of the tables below,

$$\lim_{x\to 0} \frac{7^x - 1}{x} \approx 1.946 < 2$$

while from the second of the tables below, we see that

$$\lim_{x\to 0} \frac{8^x - 1}{x} \approx 2.079 > 2$$

x	$\frac{7^x-1}{x}$	x	$\frac{7^x-1}{x}$
0.01	1.964966	-0.01	1.927100
0.001	1.947805	-0.001	1.944018
0.0001	1.946099	-0.0001	1.945721
0.00001	1.945929	-0.00001	1.945891

x	$\frac{8^x-1}{x}$	x	$\frac{8^x-1}{x}$
0.01	2.101213	-0.01	2.057970
0.001	2.081605	-0.001	2.077281
0.0001	2.079658	-0.0001	2.079225
0.00001	2.079463	-0.00001	2.079420

By trial and error, we find that

$$\lim_{x\to 0}\frac{b^x-1}{x}=2$$

for $b \approx 7.39$ (see the table below).

x	$\frac{7.39^x-1}{x}$	x	$\frac{7.39^x-1}{x}$
0.01	2.020264	−0.01	1.980258
0.001	2.002129	−0.001	1.998129
0.0001	2.000328	−0.0001	1.999928
0.00001	2.000148	−0.00001	2.000108

73. Investigate $\lim_{x\to 1}\frac{x^n-1}{x^m-1}$ for (m,n) equal to $(2,1)$, $(1,2)$, $(2,3)$, and $(3,2)$. Then guess the value of the limit in general and check your guess for two additional pairs.

SOLUTION

x	0.99	0.9999	1.0001	1.01
$\frac{x-1}{x^2-1}$	0.502513	0.500025	0.499975	0.497512

The limit as $x \to 1$ is $\frac{1}{2}$.

x	0.99	0.9999	1.0001	1.01
$\frac{x^2-1}{x-1}$	1.99	1.9999	2.0001	2.01

The limit as $x \to 1$ is 2.

x	0.99	0.9999	1.0001	1.01
$\frac{x^2-1}{x^3-1}$	0.670011	0.666700	0.666633	0.663344

The limit as $x \to 1$ is $\frac{2}{3}$.

x	0.99	0.9999	1.0001	1.01
$\frac{x^3-1}{x^2-1}$	1.492513	1.499925	1.500075	1.507512

The limit as $x \to 1$ is $\frac{3}{2}$.

- For general m and n, we have $\lim_{x\to 1}\frac{x^n-1}{x^m-1}=\frac{n}{m}$.

x	0.99	0.9999	1.0001	1.01
$\frac{x-1}{x^3-1}$	0.336689	0.333367	0.333300	0.330022

The limit as $x \to 1$ is $\frac{1}{3}$.

x	0.99	0.9999	1.0001	1.01
$\frac{x^3-1}{x-1}$	2.9701	2.9997	3.0003	3.0301

The limit as $x \to 1$ is 3.

x	0.99	0.9999	1.0001	1.01
$\frac{x^3-1}{x^7-1}$	0.437200	0.428657	0.428486	0.420058

The limit as $x \to 1$ is $\frac{3}{7} \approx 0.428571$.

74. Find by numerical experimentation the positive integers k such that $\lim_{x\to 0} \frac{\sin(\sin^2 x)}{x^k}$ exists.

SOLUTION

- For $k = 1$, we have $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin(\sin^2 x)}{x} = 0$.

x	−0.01	−0.0001	0.0001	0.01
$f(x)$	−0.01	−0.0001	0.0001	0.01

- For $k = 2$, we have $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin(\sin^2 x)}{x^2} = 1$.

x	−0.01	−0.0001	0.0001	0.01
$f(x)$	0.999967	1.000000	1.000000	0.999967

- For $k = 3$, the limit does not exist.

x	−0.01	−0.0001	0.0001	0.01
$f(x)$	-10^2	-10^4	10^4	10^2

Indeed, as $x \to 0^-$, $f(x) = \frac{\sin(\sin^2 x)}{x^3} \to -\infty$, whereas as $x \to 0^+$, $f(x) = \frac{\sin(\sin^2 x)}{x^3} \to \infty$.

- For $k = 4$, we have $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin(\sin^2 x)}{x^4} = \infty$.

x	−0.01	−0.0001	0.0001	0.01
$f(x)$	10^4	10^8	10^8	10^4

- For $k = 5$, the limit does not exist.

x	−0.01	−0.0001	0.0001	0.01
$f(x)$	-10^6	-10^{12}	10^{12}	10^6

Indeed, as $x \to 0^-$, $f(x) = \frac{\sin(\sin^2 x)}{x^5} \to -\infty$, whereas as $x \to 0^+$, $f(x) = \frac{\sin(\sin^2 x)}{x^5} \to \infty$.

- For $k = 6$, we have $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin(\sin^2 x)}{x^6} = \infty$.

x	−0.01	−0.0001	0.0001	0.01
$f(x)$	10^8	10^{16}	10^{16}	10^8

- SUMMARY
  - For $k = 1$, the limit is 0.
  - For $k = 2$, the limit is 1.
  - For odd $k > 2$, the limit does not exist.
  - For even $k > 2$, the limit is ∞.

75. GU Plot the graph of $f(x) = \frac{2^x - 8}{x - 3}$.

(a) Zoom in on the graph to estimate $L = \lim_{x\to 3} f(x)$.

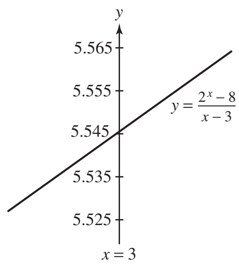
(b) Explain why

$$f(2.99999) \le L \le f(3.00001)$$

Use this to determine L to three decimal places.

SOLUTION

(a)



(b) It is clear that the graph of f rises as we move to the right. Mathematically, we may express this observation as: whenever $u < v$, $f(u) < f(v)$. Because

$$2.99999 < 3 = \lim_{x\to 3} x < 3.00001$$

it follows that

$$f(2.99999) < L = \lim_{x\to 3} f(x) < f(3.00001)$$

With $f(2.99999) \approx 5.54516$ and $f(3.00001) \approx 5.545195$, the above inequality becomes $5.54516 < L < 5.545195$; hence, to three decimal places, $L = 5.545$.

76. GU The function $f(x) = \dfrac{2^{1/x} - 2^{-1/x}}{2^{1/x} + 2^{-1/x}}$ is defined for $x \neq 0$.

(a) Investigate $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^-} f(x)$ numerically.

(b) Plot the graph of f and describe its behavior near $x = 0$.

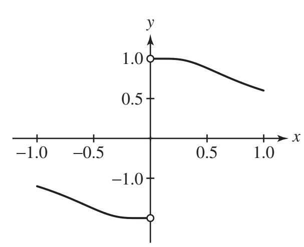
SOLUTION

(a)

x	−0.3	−0.2	−0.1	0.1	0.2	0.3
$f(x)$	−0.980506	−0.998049	−0.999998	0.999998	0.998049	0.980506

It appears that $\lim_{x\to 0^+} f(x) = 1$, while $\lim_{x\to 0^-} f(x) = -1$.

(b) As $x \to 0^-$, $f(x) \to -1$, whereas as $x \to 0^+$, $f(x) \to 1$.



2.3 Basic Limit Laws

Preliminary Questions

1. State the Sum Law and Quotient Law.

SOLUTION Suppose $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist. The Sum Law states that

$$\lim_{x\to c}(f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$$

Provided $\lim_{x\to c} g(x) \neq 0$, the Quotient Law states that

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$$

2. Which of the following is a verbal version of the Product Law (assuming the limits exist)?
(a) The product of two functions has a limit.
(b) The limit of the product is the product of the limits.
(c) The product of a limit is a product of functions.
(d) A limit produces a product of functions.

SOLUTION The verbal version of the Product Law is **(b)**: The limit of the product is the product of the limits.

3. Which statement is correct? The Quotient Law does not hold if
(a) The limit of the denominator is zero
(b) The limit of the numerator is zero

SOLUTION Statement **(a)** is correct.

Exercises

In Exercises 1–26, evaluate the limit using the Basic Limit Laws and the limits $\lim_{x\to c} x^{p/q} = c^{p/q}$ *and* $\lim_{x\to c} k = k$.

1. $\lim_{x\to 9} x$

SOLUTION $\lim_{x\to 9} x = 9$

2. $\lim_{x\to -3} 14$

SOLUTION $\lim_{x\to -3} 14 = 14$

3. $\lim_{x\to \frac{1}{2}} x^4$

SOLUTION $\lim_{x\to \frac{1}{2}} x^4 = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$

4. $\lim_{z\to 27} z^{2/3}$

SOLUTION $\lim_{z\to 27} z^{2/3} = 27^{2/3} = 81$

5. $\lim_{t\to 2} t^{-1}$

SOLUTION We apply the definition of t^{-1}, and then the Quotient Law:

$$\lim_{t\to 2} t^{-1} = \lim_{t\to 2} \frac{1}{t} = \frac{\lim_{t\to 2} 1}{\lim_{t\to 2} t} = \frac{1}{2}$$

6. $\lim_{x\to 5} x^{-2}$

SOLUTION We apply the definition of $x^{-2} = \frac{1}{x^2}$, and then the Quotient Law and the Law for Powers:

$$\lim_{x\to 5} x^{-2} = \frac{\lim_{x\to 5} 1}{\lim_{x\to 5} x^2} = \frac{1}{5^2} = \frac{1}{25}$$

7. $\lim_{x\to 0.2} (3x + 4)$

SOLUTION We apply the Laws for Sums and Constant multiples:

$$\begin{aligned}\lim_{x\to 0.2} (3x + 4) &= \lim_{x\to 0.2} 3x + \lim_{x\to 0.2} 4 \\ &= 3 \lim_{x\to 0.2} x + \lim_{x\to 0.2} 4 = 3(0.2) + 4 = 4.6\end{aligned}$$

8. $\lim_{x\to \frac{1}{3}} (3x^3 + 2x^2)$

SOLUTION We apply the Laws for Sums, Constant multiples, and Powers:

$$\begin{aligned}\lim_{x\to \frac{1}{3}} (3x^3 + 2x^2) &= \lim_{x\to \frac{1}{3}} 3x^3 + \lim_{x\to \frac{1}{3}} 2x^2 \\ &= 3 \lim_{x\to \frac{1}{3}} x^3 + 2 \lim_{x\to \frac{1}{3}} x^2 \\ &= 3\left(\frac{1}{3}\right)^3 + 2\left(\frac{1}{3}\right)^2 = \frac{1}{9} + \frac{2}{9} = \frac{1}{3}\end{aligned}$$

9. $\lim_{x\to -1}(3x^4 - 2x^3 + 4x)$

SOLUTION We apply the Laws for Sums, Constant multiples, and Powers:

$$\begin{aligned}\lim_{x\to -1}(3x^4 - 2x^3 + 4x) &= \lim_{x\to -1} 3x^4 - \lim_{x\to -1} 2x^3 + \lim_{x\to -1} 4x \\ &= 3\lim_{x\to -1} x^4 - 2\lim_{x\to -1} x^3 + 4\lim_{x\to -1} x \\ &= 3(-1^4) - 2(-1^3) - 4 = 3 + 2 - 4 = 1\end{aligned}$$

10. $\lim_{x\to 8}(3x^{2/3} - 16x^{-1})$

SOLUTION We apply the Laws for Sums, Constant multiples, and Powers and roots:

$$\begin{aligned}\lim_{x\to 8}(3x^{2/3} - 16x^{-1}) &= \lim_{x\to 8} 3x^{2/3} + \lim_{x\to 8} 16x^{-1} \\ &= 3\lim_{x\to 8} x^{2/3} + 16\lim_{x\to 8} x^{-1} \\ &= 3(8)^{2/3} + 16(8)^{-1} = 12 + 2 = 14\end{aligned}$$

11. $\lim_{x\to 2}(x + 1)(3x^2 - 9)$

SOLUTION We apply the Laws for Products, Sums, Constant multiples, and Powers:

$$\begin{aligned}\lim_{x\to 2}(x + 1)\left(3x^2 - 9\right) &= \left(\lim_{x\to 2} x + \lim_{x\to 2} 1\right)\left(3\lim_{x\to 2} x^2 - \lim_{x\to 2} 9\right) \\ &= (2 + 1)\left(3(2)^2 - 9\right) = 3 \cdot 3 = 9\end{aligned}$$

12. $\lim_{x\to \frac{1}{2}}(4x + 1)(6x - 1)$

SOLUTION We apply the Laws for Products, Sums, and Constant multiples:

$$\begin{aligned}\lim_{x\to 1/2}(4x + 1)(6x - 1) &= \left(4\lim_{x\to 1/2} x + \lim_{x\to 1/2} 1\right)\left(6\lim_{x\to 1/2} x - \lim_{x\to 1/2} 1\right) \\ &= \left(4\left(\frac{1}{2}\right) + 1\right)\left(6\left(\frac{1}{2}\right) - 1\right) = 3 \cdot 2 = 6\end{aligned}$$

13. $\lim_{t\to 4}\frac{1}{t + 4}$

SOLUTION We apply the Laws for Quotients and Sums:

$$\lim_{t\to 4}\frac{1}{t + 4} = \frac{\lim_{t\to 4} 1}{\lim_{t\to 4} t + 4} = \frac{1}{4 + 4} = \frac{1}{8}$$

14. $\lim_{z\to 0}\frac{3}{z - 1}$

SOLUTION We apply the Laws for Quotients and Sums:

$$\lim_{z\to 0}\frac{3}{z - 1} = \frac{\lim_{z\to 0} 3}{\lim_{z\to 0} z - 1} = \frac{3}{0 - 1} = -3$$

15. $\lim_{t\to 4}\frac{3t - 14}{t + 1}$

SOLUTION We apply the Laws for Quotients, Sums, and Constant multiples:

$$\lim_{t\to 4}\frac{3t - 14}{t + 1} = \frac{3\lim_{t\to 4} t - \lim_{t\to 4} 14}{\lim_{t\to 4} t + \lim_{t\to 4} 1} = \frac{3 \cdot 4 - 14}{4 + 1} = -\frac{2}{5}$$

16. $\lim_{z\to 9} \frac{\sqrt{z}}{z-2}$

SOLUTION We apply the Laws for Quotients, Roots, and Sums:

$$\lim_{z\to 9} \frac{\sqrt{z}}{z-2} = \frac{\lim_{z\to 9} \sqrt{z}}{\lim_{z\to 9} z - 2} = \frac{\sqrt{9}}{9-2} = \frac{3}{7}$$

17. $\lim_{y\to \frac{1}{4}} (16y+1)(2y^{1/2}+1)$

SOLUTION We apply the Laws for Products, Sums, Constant multiples, and Roots:

$$\lim_{y\to \frac{1}{4}} (16y+1)(2y^{1/2}+1) = \left(16 \lim_{y\to \frac{1}{4}} y + \lim_{y\to \frac{1}{4}} 1\right) \cdot \left(2 \lim_{y\to \frac{1}{4}} y^{1/2} + \lim_{y\to \frac{1}{4}} 1\right)$$

$$= \left(16\left(\frac{1}{4}\right) + 1\right) \cdot \left(2\left(\frac{1}{4}\right)^{1/2} + 1\right) = 5 \cdot 2 = 10$$

18. $\lim_{x\to 2} x(x+1)(x+2)$

SOLUTION We apply the Product Law and Sum Law:

$$\lim_{x\to 2} x(x+1)(x+2) = \left(\lim_{x\to 2} x\right)\left(\lim_{x\to 2}(x+1)\right)\left(\lim_{x\to 2}(x+2)\right)$$

$$= 2\left(\lim_{x\to 2} x + \lim_{x\to 2} 1\right)\left(\lim_{x\to 2} x + \lim_{x\to 2} 2\right)$$

$$= 2(2+1)(2+2) = 24$$

19. $\lim_{y\to 4} \frac{1}{\sqrt{6y+1}}$

SOLUTION We apply the Laws for Quotients, Roots, Sums, and Constant multiples:

$$\lim_{y\to 4} \frac{1}{\sqrt{6y+1}} = \frac{\lim_{y\to 4} 1}{\sqrt{6 \lim_{y\to 4} y + \lim_{y\to 4} 1}} = \frac{1}{\sqrt{6(4)+1}} = \frac{1}{\sqrt{25}} = \frac{1}{5}$$

20. $\lim_{w\to 7} \frac{\sqrt{w+2}+1}{\sqrt{w-3}-1}$

SOLUTION We apply the Laws for Quotients, Sums, and Roots:

$$\lim_{w\to 7} \frac{\sqrt{w+2}+1}{\sqrt{w-3}-1} = \frac{\sqrt{\lim_{w\to 7} w + \lim_{w\to 7} 2} + \lim_{w\to 7} 1}{\sqrt{\lim_{w\to 7} w - \lim_{w\to 7} 3} - \lim_{w\to 7} 1} = \frac{\sqrt{7+2}+1}{\sqrt{7-3}-1} = 4$$

21. $\lim_{x\to -1} \frac{x}{x^3+4x}$

SOLUTION We apply the Laws for Quotients, Sums, Powers, and Constant multiples:

$$\lim_{x\to -1} \frac{x}{x^3+4x} = \frac{\lim_{x\to -1} x}{\lim_{x\to -1} x^3 + 4 \lim_{x\to -1} x} = \frac{-1}{(-1)^3+4(-1)} = \frac{-1}{-1-4} = \frac{1}{5}$$

22. $\lim_{t\to -1} \frac{t^2+1}{(t^3+2)(t^4+1)}$

SOLUTION We apply the Laws for Quotients, Products, Sums, and Powers:

$$\lim_{t\to -1} \frac{t^2+1}{(t^3+2)(t^4+1)} = \frac{\lim_{t\to -1} t^2 + \lim_{t\to -1} 1}{\left(\lim_{t\to -1} t^3 + \lim_{t\to -1} 2\right) \cdot \left(\lim_{t\to -1} t^4 + \lim_{t\to -1} 1\right)}$$

$$= \frac{(-1)^2+1}{((-1)^3+2)((-1)^4+1)} = \frac{2}{1(2)} = 1$$

23. $\displaystyle\lim_{t\to 25} \frac{3\sqrt{t} - \frac{1}{5}t}{(t-20)^2}$

SOLUTION We apply the Laws for Quotients, Sums, Constant multiples, and Powers and Roots:

$$\lim_{t\to 25} \frac{3\sqrt{t} - \frac{1}{5}t}{(t-20)^2} = \frac{3\lim_{t\to 25}\sqrt{t} - \dfrac{1}{5}\lim_{t\to 25} t}{\left(\lim_{t\to 25} t - \lim_{t\to 25} 20\right)^2} = \frac{3\sqrt{25} - \frac{1}{5}(25)}{(25-20)^2} = \frac{10}{25} = \frac{2}{5}$$

24. $\displaystyle\lim_{y\to \frac{1}{3}} (18y^2 - 4)^4$

SOLUTION We apply the Laws for Powers, Sums, and Constant multiples:

$$\lim_{y\to \frac{1}{3}} (18y^2 - 4)^4 = \left(18\left(\frac{1}{3}\right)^2 - 4\right)^4 = (-2)^4 = 16$$

25. $\displaystyle\lim_{t\to \frac{3}{2}} (4t^2 + 8t - 5)^{3/2}$

SOLUTION We apply the Laws for Powers, Sums, and Constant multiples:

$$\lim_{t\to \frac{3}{2}} (4t^2 + 8t - 5)^{3/2} = \left(4\left(\frac{3}{2}\right)^2 + 8\cdot\frac{3}{2} - 5\right)^{3/2} = 16^{3/2} = 64$$

26. $\displaystyle\lim_{t\to 7} \frac{(t+2)^{1/2}}{(t+1)^{2/3}}$

SOLUTION We apply the Laws for Quotients, Roots, and Sums:

$$\lim_{t\to 7} \frac{(t+2)^{1/2}}{(t+1)^{2/3}} = \frac{\lim_{t\to 7}(t+2)^{1/2}}{\lim_{t\to 7}(t+1)^{2/3}} = \frac{9^{1/2}}{8^{2/3}} = \frac{3}{4}$$

27. Use the Quotient Law to prove that if $\lim_{x\to c} f(x)$ exists and is nonzero, then

$$\lim_{x\to c} \frac{1}{f(x)} = \frac{1}{\lim_{x\to c} f(x)}$$

SOLUTION Since $\lim_{x\to c} f(x)$ is nonzero, we can apply the Quotient Law:

$$\lim_{x\to c} \left(\frac{1}{f(x)}\right) = \frac{\left(\lim_{x\to c} 1\right)}{\left(\lim_{x\to c} f(x)\right)} = \frac{1}{\lim_{x\to c} f(x)}$$

28. Assuming that $\lim_{x\to 6} f(x) = 4$, compute:

(a) $\displaystyle\lim_{x\to 6} f(x)^2$ **(b)** $\displaystyle\lim_{x\to 6} \frac{1}{f(x)}$ **(c)** $\displaystyle\lim_{x\to 6} x\sqrt{f(x)}$

SOLUTION

(a) Apply the Product Law:

$$\lim_{x\to 6} f(x)^2 = \left(\lim_{x\to 6} f(x)\right)\left(\lim_{x\to 6} f(x)\right) = (4)(4) = 16$$

(b) Since $\lim_{x\to 6} f(x) \neq 0$, we may apply the Quotient Law:

$$\lim_{x\to 6} \frac{1}{f(x)} = \frac{1}{\lim_{x\to 6} f(x)} = \frac{1}{4}$$

(c) Apply the Product Law and the Law for Roots:

$$\lim_{x\to 6} x\sqrt{f(x)} = \left(\lim_{x\to 6} x\right)\left(\lim_{x\to 6} f(x)\right)^{1/2} = 6(4)^{1/2} = 12$$

In Exercises 29–32, evaluate the limit assuming that $\lim_{x\to -4} f(x) = 3$ *and* $\lim_{x\to -4} g(x) = 1$.

29. $\lim_{x\to -4} f(x)g(x)$

SOLUTION $\lim_{x\to -4} f(x)g(x) = \lim_{x\to -4} f(x) \lim_{x\to -4} g(x) = 3 \cdot 1 = 3$

30. $\lim_{x\to -4} (2f(x) + 3g(x))$

SOLUTION

$$\begin{aligned}\lim_{x\to -4} (2f(x) + 3g(x)) &= 2 \lim_{x\to -4} f(x) + 3 \lim_{x\to -4} g(x) \\ &= 2 \cdot 3 + 3 \cdot 1 = 6 + 3 = 9\end{aligned}$$

31. $\lim_{x\to -4} \frac{g(x)}{x^2}$

SOLUTION Since $\lim_{x\to -4} x^2 \neq 0$, we may apply the Quotient Law, followed by the Law for Powers:

$$\lim_{x\to -4} \frac{g(x)}{x^2} = \frac{\lim_{x\to -4} g(x)}{\lim_{x\to -4} x^2} = \frac{1}{(-4)^2} = \frac{1}{16}$$

32. $\lim_{x\to -4} \frac{f(x)+1}{3g(x)-9}$

SOLUTION

$$\lim_{x\to -4} \frac{f(x)+1}{3g(x)-9} = \frac{\lim_{x\to -4} f(x) + \lim_{x\to -4} 1}{3 \lim_{x\to -4} g(x) - \lim_{x\to -4} 9} = \frac{3+1}{3 \cdot 1 - 9} = \frac{4}{-6} = -\frac{2}{3}$$

33. Can the Quotient Law be applied to evaluate $\lim_{x\to 0} \frac{\sin x}{x}$? Explain.

SOLUTION The limit Quotient Law *cannot* be applied to evaluate $\lim_{x\to 0} \frac{\sin x}{x}$ since $\lim_{x\to 0} x = 0$. This violates a condition of the Quotient Law. Accordingly, the rule *cannot* be employed.

34. Show that the Product Law cannot be used to evaluate the limit $\lim_{x\to \pi/2} (x - \frac{\pi}{2}) \tan x$.

SOLUTION The limit Product Law *cannot* be applied to evaluate $\lim_{x\to \pi/2} (x - \pi/2) \tan x$ since $\lim_{x\to \pi/2} \tan x$ does not exist (for example, as $x \to \pi/2-$, $\tan x \to \infty$). This violates a hypothesis of the Product Law. Accordingly, the rule *cannot* be employed.

35. Assume that if $\lim_{x\to a} f(x) = L$, then $\lim_{x\to a} \sin f(x) = \sin L$. In each case evaluate the limit or indicate that the limit does not exist.

(a) $\lim_{x\to 0} \sin\left(\frac{x}{x-1}\right)$

(b) $\lim_{x\to \pi/2} \frac{\sin x}{x}$

(c) $\lim_{x\to 1} \frac{3x}{\sin(1-x)}$

(d) $\lim_{x\to 1} x^2 \sin(\pi x^2)$

SOLUTION

(a) Because $\lim_{x\to 0} \frac{x}{x-1} = \frac{0}{0-1} = 0$, it follows that

$$\lim_{x\to 0} \sin\left(\frac{x}{x-1}\right) = \sin 0 = 0$$

(b) By the Quotient Law,

$$\lim_{x\to \pi/2} \frac{\sin x}{x} = \frac{\sin \frac{\pi}{2}}{\frac{\pi}{2}} = \frac{2}{\pi}$$

(c) Because $\lim_{x\to 1}(1 - x) = 1 - 1 = 0$, it follows that

$$\lim_{x\to 0} \sin(1 - x) = \sin 0 = 0$$

Now, $\lim_{x\to 1} 3x = 3(1) = 3 \neq 0$, so

$$\lim_{x\to 1} \frac{3x}{\sin(1-x)} \text{ does not exist}$$

(d) Because $\lim_{x\to 1} \pi x^2 = \pi(1)^2 = \pi$, it follows that

$$\lim_{x\to 1} \sin(\pi x^2) = \sin \pi = 0$$

Then, by the Product Law,

$$\lim_{x\to 1} x^2 \sin(\pi x^2) = \lim_{x\to 1} x^2 \cdot \lim_{x\to 1} \sin(\pi x^2) = 1^2 \cdot 0 = 0$$

36. Assume that if $\lim_{x\to a} f(x) = L$, then $\lim_{x\to a} \cos f(x) = \cos L$. In each case evaluate the limit or indicate that the limit does not exist.

(a) $\lim_{x\to 0} \cos\left(\frac{2x}{1-2x}\right)$

(b) $\lim_{x\to \pi/2} \frac{\cos x}{x}$

(c) $\lim_{x\to 1} x^3 \cos(1-x)$

(d) $\lim_{x\to 0} \frac{1-x^2}{1-\cos(x^2)}$

SOLUTION

(a) Because $\lim_{x\to 0} \frac{2x}{1-2x} = \frac{2(0)}{1-2(0)} = 0$, it follows that

$$\lim_{x\to 0} \cos\left(\frac{2x}{1-2x}\right) = \cos 0 = 1$$

(b) By the Quotient Law,

$$\lim_{x\to \pi/2} \frac{\cos x}{x} = \frac{\cos \frac{\pi}{2}}{\frac{\pi}{2}} = 0$$

(c) Because $\lim_{x\to 1}(1-x) = 1-1 = 0$, it follows that

$$\lim_{x\to 0} \cos(1-x) = \cos 0 = 1$$

Then, by the Product Law,

$$\lim_{x\to 1} x^3 \cos(1-x) = \lim_{x\ to 1} x^3 \cdot \lim_{x\to 1} \cos(1-x) = 1^3 \cdot 1 = 1$$

(d) Because $\lim_{x\to 0} x^2 = 0^2 = 0$, it follows that

$$\lim_{x\to 0} \cos(x^2) = \cos 0 = 1 \quad \text{and} \quad \lim_{x\to 0}(1-\cos(x^2)) = 1-1 = 0$$

Now, $\lim_{x\to 0}(1-x^2) = 1-0^2 = 1 \neq 0$, so

$$\lim_{x\to 0} \frac{1-x^2}{1-\cos(x^2)} \text{ does not exist}$$

37. Give an example where $\lim_{x\to 0}(f(x)+g(x))$ exists but neither $\lim_{x\to 0} f(x)$ nor $\lim_{x\to 0} g(x)$ exists.

SOLUTION Let $f(x) = 1/x$ and $g(x) = -1/x$. Then $\lim_{x\to 0} (f(x)+g(x)) = \lim_{x\to 0} 0 = 0$. However, $\lim_{x\to 0} f(x) = \lim_{x\to 0} 1/x$ and $\lim_{x\to 0} g(x) = \lim_{x\to 0} -1/x$ do not exist.

38. Give an example where $\lim_{x\to 0}(f(x)\cdot g(x))$ exists but neither $\lim_{x\to 0} f(x)$ nor $\lim_{x\to 0} g(x)$ exists.

SOLUTION Let

$$f(x) = \begin{cases} -1, & x < 0 \\ 1, & x \geq 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 1, & x < 0 \\ -1, & x \geq 0 \end{cases}$$

Then $\lim_{x\to 0}(f(x)\cdot g(x)) = \lim_{x\to 0}(-1) = -1$; however, neither $\lim_{x\to 0} f(x)$ nor $\lim_{x\to 0} g(x)$ exists.

39. Give an example where $\lim_{x\to 0} \frac{f(x)}{g(x)}$ exists but neither $\lim_{x\to 0} f(x)$ nor $\lim_{x\to 0} g(x)$ exists.

SOLUTION Let

$$f(x) = \begin{cases} -1, & x < 0 \\ 1, & x \geq 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 1, & x < 0 \\ -1, & x \geq 0 \end{cases}$$

Then $\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0}(-1) = -1$; however, neither $\lim_{x\to 0} f(x)$ nor $\lim_{x\to 0} g(x)$ exists.

Further Insights and Challenges

40. Show that if both $\lim_{x\to c} f(x)\,g(x)$ and $\lim_{x\to c} g(x)$ exist and $\lim_{x\to c} g(x) \neq 0$, then $\lim_{x\to c} f(x)$ exists. *Hint:* Write $f(x) = \dfrac{f(x)\,g(x)}{g(x)}$.

SOLUTION Given that $\lim_{x\to c} f(x)g(x) = L$ and $\lim_{x\to c} g(x) = M \neq 0$ both exist, observe that

$$\lim_{x\to c} f(x) = \lim_{x\to c} \frac{f(x)g(x)}{g(x)} = \frac{\lim_{x\to c} f(x)g(x)}{\lim_{x\to c} g(x)} = \frac{L}{M}$$

also exists.

41. Suppose that $\lim_{t\to 3} tg(t) = 12$. Show that $\lim_{t\to 3} g(t)$ exists and equals 4.

SOLUTION We are given that $\lim_{t\to 3} tg(t) = 12$. Since $\lim_{t\to 3} t = 3 \neq 0$, we may apply the Quotient Law:

$$\lim_{t\to 3} g(t) = \lim_{t\to 3} \frac{tg(t)}{t} = \frac{\lim_{t\to 3} tg(t)}{\lim_{t\to 3} t} = \frac{12}{3} = 4$$

42. Prove that if $\lim_{t\to 3} \frac{h(t)}{t} = 5$, then $\lim_{t\to 3} h(t) = 15$.

SOLUTION Given that $\lim_{t\to 3} \dfrac{h(t)}{t} = 5$, observe that $\lim_{t\to 3} t = 3$. Now use the Product Law:

$$\lim_{t\to 3} h(t) = \lim_{t\to 3} t\,\frac{h(t)}{t} = \left(\lim_{t\to 3} t\right)\left(\lim_{t\to 3} \frac{h(t)}{t}\right) = 3\cdot 5 = 15$$

43. Assuming that $\lim_{x\to 0} \frac{f(x)}{x} = 1$, which of the following statements is necessarily true? Why?

(a) $f(0) = 0$

(b) $\lim_{x\to 0} f(x) = 0$

SOLUTION

(a) Given that $\lim_{x\to 0} \dfrac{f(x)}{x} = 1$, it is not necessarily true that $f(0) = 0$. A counterexample is provided by $f(x) = \begin{cases} x, & x \neq 0 \\ 5, & x = 0 \end{cases}$.

(b) Given that $\lim_{x\to 0} \dfrac{f(x)}{x} = 1$, it is necessarily true that $\lim_{x\to 0} f(x) = 0$. For note that $\lim_{x\to 0} x = 0$, whence

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x\,\frac{f(x)}{x} = \left(\lim_{x\to 0} x\right)\left(\lim_{x\to 0} \frac{f(x)}{x}\right) = 0\cdot 1 = 0$$

44. Prove that if $\lim_{x\to c} f(x) = L \neq 0$ and $\lim_{x\to c} g(x) = 0$, then the limit $\lim_{x\to c} \frac{f(x)}{g(x)}$ does not exist.

SOLUTION Suppose that $\lim_{x\to c} \dfrac{f(x)}{g(x)}$ exists. Then

$$L = \lim_{x\to c} f(x) = \lim_{x\to c} g(x)\cdot\frac{f(x)}{g(x)} = \lim_{x\to c} g(x)\cdot \lim_{x\to c} \frac{f(x)}{g(x)} = 0\cdot \lim_{x\to c} \frac{f(x)}{g(x)} = 0$$

But, we were given that $L \neq 0$, so we have arrived at a contradiction. Thus, $\lim_{x\to c} \dfrac{f(x)}{g(x)}$ does not exist.

45. Suppose that $\lim_{h\to 0} g(h) = L$.

(a) Explain why $\lim_{h\to 0} g(ah) = L$ for any constant $a \neq 0$.

(b) If we assume instead that $\lim_{h\to 1} g(h) = L$, is it still necessarily true that $\lim_{h\to 1} g(ah) = L$?

(c) Illustrate (a) and (b) with the function $f(x) = x^2$.

SOLUTION

(a) As $h \to 0$, $ah \to 0$ as well; hence, if we make the change of variable $w = ah$, then

$$\lim_{h\to 0} g(ah) = \lim_{w\to 0} g(w) = L$$

(b) No. As $h \to 1$, $ah \to a$, so we should not expect $\lim_{h\to 1} g(ah) = \lim_{h\to 1} g(h)$.

(c) Let $g(x) = x^2$. Then

$$\lim_{h\to 0} g(h) = 0 \quad \text{and} \quad \lim_{h\to 0} g(ah) = \lim_{h\to 0}(ah)^2 = 0$$

On the other hand,

$$\lim_{h\to 1} g(h) = 1 \quad \text{while} \quad \lim_{h\to 1} g(ah) = \lim_{h\to 1}(ah)^2 = a^2$$

which is equal to the previous limit if and only if $a = \pm 1$.

46. Assume that $L(a) = \lim_{x\to 0} \frac{a^x - 1}{x}$ exists for all $a > 0$. Assume also that $\lim_{x\to 0} a^x = 1$.
(a) Prove that $L(ab) = L(a) + L(b)$ for $a, b > 0$. *Hint:* $(ab)^x - 1 = a^x b^x - a^x + a^x - 1 = a^x(b^x - 1) + (a^x - 1)$. [This shows that $L(a)$ "behaves" like a logarithm, in the sense that $\log(ab) = \log(a) + \log(b)$. In fact, it can be shown that $L(a)$ is equal to what is known as the natural logarithm function.]
(b) Verify numerically that $L(12) = L(3) + L(4)$.

SOLUTION
(a) Let $a, b > 0$. Then

$$\begin{aligned} L(ab) = \lim_{x\to 0} \frac{(ab)^x - 1}{x} &= \lim_{x\to 0} \frac{a^x(b^x - 1) + (a^x - 1)}{x} \\ &= \lim_{x\to 0} a^x \cdot \lim_{x\to 0} \frac{b^x - 1}{x} + \lim_{x\to 0} \frac{a^x - 1}{x} \\ &= 1 \cdot L(b) + L(a) = L(a) + L(b) \end{aligned}$$

(b) From the table below, we estimate that, to three decimal places, $L(3) = 1.099$, $L(4) = 1.386$, and $L(12) = 2.485$. Thus,

$$L(12) = 2.485 = 1.099 + 1.386 = L(3) + L(4)$$

x	−0.01	−0.001	−0.0001	0.0001	0.001	0.01
$(3^x - 1)/x$	1.092600	1.098009	1.098552	1.098673	1.099216	1.104669
$(4^x - 1)/x$	1.376730	1.385334	1.386198	1.386390	1.387256	1.395948
$(12^x - 1)/x$	2.454287	2.481822	2.484600	2.485215	2.488000	2.516038

2.4 Limits and Continuity

Preliminary Questions

1. Which property of $f(x) = x^3$ allows us to conclude that $\lim_{x\to 2} x^3 = 8$?

SOLUTION We can conclude that $\lim_{x\to 2} x^3 = 8$ because the function x^3 is continuous at $x = 2$.

2. What can be said about $f(3)$ if f is continuous and $\lim_{x\to 3} f(x) = \frac{1}{2}$?

SOLUTION If f is continuous and $\lim_{x\to 3} f(x) = \frac{1}{2}$, then $f(3) = \frac{1}{2}$.

3. Suppose that $f(x) < 0$ if x is positive and $f(x) > 1$ if x is negative. Can f be continuous at $x = 0$?

SOLUTION Since $f(x) < 0$ when x is positive and $f(x) > 1$ when x is negative, it follows that

$$\lim_{x\to 0^+} f(x) \le 0 \quad \text{and} \quad \lim_{x\to 0^-} f(x) \ge 1$$

Thus, $\lim_{x\to 0} f(x)$ does not exist, so f cannot be continuous at $x = 0$.

4. Is it possible to determine $f(7)$ if $f(x) = 3$ for all $x < 7$ and f is right-continuous at $x = 7$? What if f is left-continuous?

SOLUTION No. To determine $f(7)$, we need to combine either knowledge of the values of $f(x)$ for $x < 7$ with *left*-continuity or knowledge of the values of $f(x)$ for $x > 7$ with right-continuity.

5. Are the following true or false? If false, then draw or give a counterexample, and state a correct version.

(a) f is continuous at $x = a$ if the left- and right-hand limits of $f(x)$ as $x \to a$ exist and are equal.

(b) f is continuous at $x = a$ if the left- and right-hand limits of $f(x)$ as $x \to a$ exist and equal $f(a)$.

(c) If the left- and right-hand limits of $f(x)$ as $x \to a$ exist, then f has a removable discontinuity at $x = a$.

(d) If f and g are continuous at $x = a$, then $f + g$ is continuous at $x = a$.

(e) If f and g are continuous at $x = a$, then f/g is continuous at $x = a$.

SOLUTION

(a) False. The correct statement is "$f(x)$ is continuous at $x = a$ if the left- and right-hand limits of $f(x)$ as $x \to a$ exist and equal $f(a)$."

(b) True.

(c) False. The correct statement is "If the left- and right-hand limits of $f(x)$ as $x \to a$ are equal but not equal to $f(a)$, then f has a removable discontinuity at $x = a$."

(d) True.

(e) False. The correct statement is "If $f(x)$ and $g(x)$ are continuous at $x = a$ and $g(a) \neq 0$, then $f(x)/g(x)$ is continuous at $x = a$."

Exercises

1. Referring to Figure 15, state whether f is left- or right-continuous (or neither) at each point of discontinuity. Does f have any removable discontinuities?

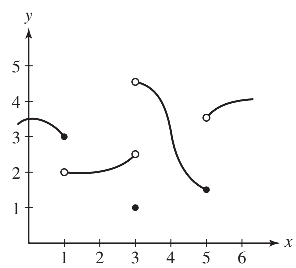


FIGURE 15 Graph of $y = f(x)$.

SOLUTION

- The function f is discontinuous at $x = 1$; it is left-continuous there.
- The function f is discontinuous at $x = 3$; it is neither left-continuous nor right-continuous there.
- The function f is discontinuous at $x = 5$; it is left-continuous there.

However, these discontinuities are not removable.

Exercises 2–4 refer to the function g whose graph appears in Figure 16.

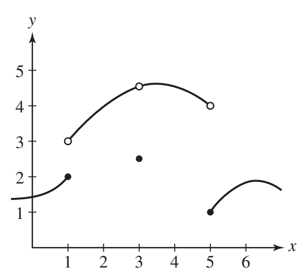


FIGURE 16 Graph of $y = g(x)$.

2. State whether g is left- or right-continuous (or neither) at each of its points of discontinuity.

SOLUTION

- The function g is discontinuous at $x = 1$; it is left-continuous there.
- The function g is discontinuous at $x = 3$; it is neither left-continuous nor right-continuous there.
- The function g is discontinuous at $x = 5$; it is right-continuous there.

3. At which point c does g have a removable discontinuity? How should $g(c)$ be redefined to make g continuous at $x = c$?

SOLUTION Because $\lim_{x \to 3} g(x)$ exists, the function g has a removable discontinuity at $x = 3$. Assigning $g(3) = 4$ makes g continuous at $x = 3$.

4. Find the point c_1 at which g has a jump discontinuity but is left-continuous. How should $g(c_1)$ be redefined to make g right-continuous at $x = c_1$?

SOLUTION The function f has a jump discontinuity at $x = 1$, but is left-continuous there. Assigning $f(1) = 3$ makes f right-continuous at $x = 1$ (but no longer left-continuous).

5. In Figure 17, determine the one-sided limits at the points of discontinuity. Which discontinuity is removable and how should f be redefined to make it continuous at this point?

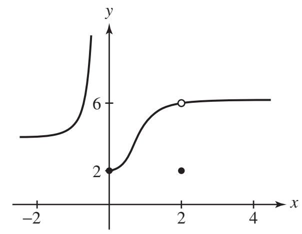


FIGURE 17

SOLUTION The function f is discontinuous at $x = 0$, at which $\lim_{x \to 0^-} f(x) = \infty$ and $\lim_{x \to 0^+} f(x) = 2$. The function f is also discontinuous at $x = 2$, at which $\lim_{x \to 2^-} f(x) = 6$ and $\lim_{x \to 2^+} f(x) = 6$. The discontinuity at $x = 2$ is removable. Assigning $f(2) = 6$ makes f continuous at $x = 2$.

6. Suppose that $f(x) = 2$ for $x < 3$ and $f(x) = -4$ for $x > 3$.

(a) What is $f(3)$ if f is left-continuous at $x = 3$?

(b) What is $f(3)$ if f is right-continuous at $x = 3$?

SOLUTION $f(x) = 2$ for $x < 3$ and $f(x) = -4$ for $x > 3$.

- If f is left-continuous at $x = 3$, then $f(3) = \lim_{x \to 3^-} f(x) = 2$.
- If f is right-continuous at $x = 3$, then $f(3) = \lim_{x \to 3^+} f(x) = -4$.

In Exercises 7–16, use Theorems 1–4 to show that the function is continuous.

7. $f(x) = x + \sin x$

SOLUTION The polynomial function x is continuous by Theorem 2, and the trigonometric function $\sin x$ is continuous by Theorem 3. Therefore, $x + \sin x$ is continuous by Theorem 1(i).

8. $f(x) = x \sin x$

SOLUTION The polynomial function x is continuous by Theorem 2, and the trigonometric function $\sin x$ is continuous by Theorem 3. Therefore, $x \sin x$ is continuous by Theorem 1(iii).

9. $f(x) = 3x + 4 \sin x$

SOLUTION The polynomial function $3x$ is continuous by Theorem 2, and the trigonometric function $\sin x$ is continuous by Theorem 3. Moreover, $4 \sin x$ is continuous by Theorem 1(ii). Therefore, $3x + 4 \sin x$ is continuous by Theorem 1(i).

10. $f(x) = 3x^3 + 8x^2 - 20x$

SOLUTION The function $f(x) = 3x^3 + 8x^2 - 20x$ is a polynomial function and is therefore continuous by Theorem 2.

11. $f(x) = \dfrac{1}{x^2 + 1}$

SOLUTION The function $f(x) = \dfrac{1}{x^2 + 1}$ is a rational function whose denominator, $x^2 + 1$, is never equal to 0. Therefore, f is continuous by Theorem 2.

12. $f(x) = \dfrac{x^2 - \cos x}{3 + \cos x}$

SOLUTION The polynomial functions x^2 and 3 are continuous by Theorem 2 and the trigonometric function $\cos x$ is continuous by Theorem 3, so $x^2 - \cos x$ and $3 + \cos x$ are continuous by Theorem 1(i). Moreover, $3 + \cos x$ is never equal to 0. Therefore, $f(x) = \dfrac{x^2 - \cos x}{3 + \cos x}$ is continuous by Theorem 1(iv).

13. $f(x) = \cos(x^2)$

SOLUTION The polynomial function x^2 is continuous by Theorem 2, and the trigonometric function $\cos x$ is continuous by Theorem 3. The function f is the composition of these two continuous functions, so f is continuous by Theorem 4.

14. $f(x) = x^{1/3}\cos 3x$

SOLUTION The function $x^{1/3}$ is continuous by Theorem 3. Moreover, the polynomial function $3x$ is continuous by Theorem 2 and the trigonometric function $\cos x$ is continuous by Theorem 3, so the composite function $\cos 3x$ is continuous by Theorem 4. Therefore, $x^{1/3}\cos 3x$ is continuous by Theorem 1(iii).

15. $f(x) = \tan\left(\frac{1}{1+x^2}\right)$

SOLUTION The function $\frac{1}{1+x^2}$ is a rational function whose denominator, $1 + x^2$, is never equal to 0, so this function is continuous by Theorem 2. Moreover, because $\frac{1}{1+x^2}$ takes values in the interval $(0, 1]$ and the trigonometric function $\tan x$ is continuous on $(0, 1]$ by Theorem 3, it follows that the composite function $\tan\left(\frac{1}{1+x^2}\right)$ is continuous by Theorem 4.

16. $f(x) = \tan\frac{\pi x^2}{1+2x^2}$

SOLUTION The function $\frac{\pi x^2}{1+2x^2}$ is a rational function whose denominator, $1 + 2x^2$, is never equal to 0, so this function is continuous by Theorem 2. Moreover, because $\frac{\pi x^2}{1+2x^2}$ takes values in the interval $[0, \pi/2)$ and the trigonometric function $\tan x$ is continuous on $[0, \pi/2)$ by Theorem 3, it follows that the composite function $\tan\frac{\pi x^2}{1+2x^2}$ is continuous by Theorem 4.

In Exercises 17–38, determine the points of discontinuity. State the type of discontinuity (removable, jump, infinite, or none of these) and whether the function is left- or right-continuous.

17. $f(x) = \dfrac{1}{x}$

SOLUTION The function $1/x$ is discontinuous at $x = 0$, at which there is an infinite discontinuity. The function is neither left-continuous nor right-continuous at $x = 0$.

18. $f(x) = |x|$

SOLUTION The function $f(x) = |x|$ is continuous everywhere.

19. $f(x) = \dfrac{x-2}{|x-1|}$

SOLUTION The function $\dfrac{x-2}{|x-1|}$ is discontinuous at $x = 1$, at which there is an infinite discontinuity. The function is neither left-continuous nor right-continuous at $x = 1$.

20. $f(x) = \lfloor x \rfloor$

SOLUTION This function has a jump discontinuity at $x = n$ for every integer n. It is continuous at all other values of x. For every integer n,

$$\lim_{x\to n^+}\lfloor x \rfloor = n = f(n)$$

since $\lfloor x \rfloor = n$ for all x between n and $n + 1$. This shows that $\lfloor x \rfloor$ is *right-continuous* at $x = n$. On the other hand,

$$\lim_{x\to n^-}\lfloor x \rfloor = n - 1 \neq f(n)$$

since $\lfloor x \rfloor = n - 1$ for all x between $n - 1$ and n. Thus $\lfloor x \rfloor$ is not left-continuous at $x = n$.

21. $f(x) = \left\lfloor \dfrac{x}{2} \right\rfloor$

SOLUTION The function $\left\lfloor \dfrac{x}{2} \right\rfloor$ is discontinuous at even integers, at which there are jump discontinuities. For every integer n,

$$\lim_{x\to 2n^+}\left\lfloor \frac{x}{2} \right\rfloor = n = f(2n)$$

so that $\left\lfloor \dfrac{x}{2} \right\rfloor$ is *right-continuous* at $x = 2n$. On the other hand,

$$\lim_{x\to 2n^-}\left\lfloor \frac{x}{2} \right\rfloor = n - 1 \neq f(2n)$$

so that $\left\lfloor \dfrac{x}{2} \right\rfloor$ is not left-continuous at $x = 2n$.

22. $g(t) = \dfrac{1}{t^2 - 1}$

SOLUTION The function $g(t) = \dfrac{1}{t^2-1} = \dfrac{1}{(t-1)(t+1)}$ is discontinuous at $t = -1$ and $t = 1$, at which there are infinite discontinuities. The function is neither left-continuous nor right-continuous at $t = \pm 1$.

23. $h(x) = \dfrac{1}{2 - |x|}$

SOLUTION The function $h(x) = \dfrac{1}{2 - |x|}$ is discontinuous at $x = 2$, at which there is an infinite discontinuity. The function is neither left-continuous nor right-continuous at $x = 2$.

24. $k(x) = \dfrac{x - 2}{|2 - x|}$

SOLUTION The function $k(x) = \dfrac{x - 2}{|2 - x|}$ is discontinuous at $x = 2$. For $x < 2$, $k(x) = \dfrac{x - 2}{2 - x} = -1$, while for $x > 2$, $k(x) = \dfrac{x - 2}{x - 2} = 1$; therefore, there is a jump discontinuity at $x = 2$. The function is neither left-continuous nor right-continuous at $x = 2$.

25. $f(x) = \dfrac{x + 1}{4x - 2}$

SOLUTION The function $f(x) = \dfrac{x + 1}{4x - 2}$ is discontinuous at $x = \frac{1}{2}$, at which there is an infinite discontinuity. The function is neither left-continuous nor right-continuous at $x = \frac{1}{2}$.

26. $h(z) = \dfrac{1 - 2z}{z^2 - z - 6}$

SOLUTION The function $h(z) = \dfrac{1 - 2z}{z^2 - z - 6} = \dfrac{1 - 2z}{(z + 2)(z - 3)}$ is discontinuous at $z = -2$ and $z = 3$, at which there are infinite discontinuities. The function is neither left-continuous nor right-continuous at $z = -2$. It is also neither left-continuous nor right-continuous at $z = 3$.

27. $f(x) = 3x^{2/3} - 9x^3$

SOLUTION The function $f(x) = 3x^{2/3} - 9x^3$ is defined and continuous for all x.

28. $g(t) = 3t^{-2/3} - 9t^3$

SOLUTION The function $g(t) = 3t^{-2/3} - 9t^3$ is discontinuous at $t = 0$, at which there is an infinite discontinuity. The function is neither left-continuous nor right-continuous at $t = 0$.

29. $f(x) = \begin{cases} \dfrac{x - 2}{|x - 2|} & x \neq 2 \\ -1 & x = 2 \end{cases}$

SOLUTION For $x > 2$, $f(x) = \dfrac{x - 2}{(x - 2)} = 1$. For $x < 2$, $f(x) = \dfrac{(x - 2)}{(2 - x)} = -1$. The function has a jump discontinuity at $x = 2$. Because

$$\lim_{x \to 2^-} f(x) = -1 = f(2)$$

the function is left-continuous at $x = 2$.

30. $f(x) = \begin{cases} \cos \dfrac{1}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$

SOLUTION As $x \to 0$, $\cos\left(\frac{1}{x}\right)$ oscillates more and more rapidly between $+1$ and -1. As such, neither

$$\lim_{x \to 0^-} \cos\left(\frac{1}{x}\right) \qquad \text{nor} \qquad \lim_{x \to 0^+} \cos\left(\frac{1}{x}\right)$$

exist. The function f is therefore discontinuous at $x = 0$. The function is neither left-continuous nor right-continuous at $x = 0$.

31. $f(x) = \frac{2x^2 - 50}{x + 5}$

SOLUTION The function $f(x) = \frac{2x^2 - 50}{x + 5}$ is discontinuous at $x = -5$. Now,

$$\lim_{x \to -5} \frac{2x^2 - 50}{x + 5} = \lim_{x \to -5} \frac{2(x - 5)(x + 5)}{x + 5} = \lim_{x \to -5} 2(x - 5) = -20$$

Because $\lim_{x \to -5} f(x)$ exists but $f(-5)$ is not defined, f has a removable discontinuity at $x = -5$. The function is neither left-continuous nor right-continuous at $x = -5$.

32. $w(t) = \frac{t + 1}{t^2 - 1}$

SOLUTION The function $w(t) = \frac{t+1}{t^2-1}$ is discontinuous at $t = \pm 1$. At $t = 1$, there is an infinite discontinuity. On the other hand, at $t = -1$, there is a removable discontinuity because $\lim_{t \to -1} w(t)$ exists. In particular,

$$\lim_{t \to -1} \frac{t+1}{t^2-1} = \lim_{t \to -1} \frac{t+1}{(t+1)(t-1)} = \lim_{t \to -1} \frac{1}{t-1} = -\frac{1}{2}$$

The function is neither left-continuous nor right-continuous at $t = \pm 1$.

33. $g(t) = \tan 2t$

SOLUTION The function $g(t) = \tan 2t = \dfrac{\sin 2t}{\cos 2t}$ is discontinuous whenever $\cos 2t = 0$; that is, whenever

$$2t = \frac{(2n+1)\pi}{2} \qquad \text{or} \qquad t = \frac{(2n+1)\pi}{4}$$

where n is an integer. At every such value of t there is an infinite discontinuity. The function is neither left-continuous nor right-continuous at any of these locations.

34. $f(x) = \csc(x^2)$

SOLUTION The function $f(x) = \csc(x^2) = \dfrac{1}{\sin(x^2)}$ is discontinuous whenever $\sin(x^2) = 0$; that is, whenever $x^2 = n\pi$ or $x = \pm\sqrt{n\pi}$, where n is a positive integer. At every such value of x there is an infinite discontinuity. The function is neither left-continuous nor right-continuous at any of these locations.

35. $f(x) = \tan(\sin x)$

SOLUTION The function $f(x) = \tan(\sin x)$ is continuous everywhere. Reason: $\sin x$ is continuous everywhere and $\tan u$ is continuous on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$—and in particular on $-1 \le u = \sin x \le 1$. Continuity of $\tan(\sin x)$ follows by the continuity of composite functions.

36. $f(x) = \cos(\pi\lfloor x \rfloor)$

SOLUTION The function $\cos(\pi\lfloor x \rfloor)$ has a jump discontinuity at $x = n$ for every integer n. For every integer n,

$$\lim_{x \to n^+} \cos(\pi\lfloor x \rfloor) = \cos(n\pi) = f(n)$$

so that f is *right-continuous* at $x = n$. On the other hand,

$$\lim_{x \to n^-} \cos(\pi\lfloor x \rfloor) = \cos((n-1)\pi) \ne f(n)$$

so that f is not left-continuous at $x = n$.

37. $f(x) = \lfloor x + 3 \rfloor + \lfloor 2x \rfloor$

SOLUTION The function $\lfloor x + 3 \rfloor + \lfloor 2x \rfloor$ has a jump discontinuity at $x = n/2$ for every integer n. If $n = 2m$ for some integer m, then

$$\lim_{x \to (n/2)^+} f(x) = \lim_{x \to m^+} f(x) = 3m + 3 = f(m) = f(n/2),$$

but

$$\lim_{x \to (n/2)^-} f(x) = \lim_{x \to m^-} f(x) = 3m + 1 \ne f(m) = f(n/2);$$

while if $n = 2m + 1$ for some integer m, then

$$\lim_{x \to (n/2)^+} f(x) = \lim_{x \to (m+1/2)^+} f(x) = 3m + 4 = f(m + 1/2) = f(n/2),$$

but

$$\lim_{x \to (n/2)^-} f(x) = \lim_{x \to (m+1/2)^-} f(x) = 3m + 3 \ne f(m + 1/2) = f(n/2).$$

Thus, f is right-continuous but not left-continuous at each discontinuity.

38. $f(x) = 2\lfloor x/2 \rfloor + 4\lfloor x/4 \rfloor$

SOLUTION The function $2\lfloor x/2 \rfloor + 4\lfloor x/4 \rfloor$ has a jump discontinuity at $x = 2n$ for every integer n. If $n = 2m$ for some integer m, then

$$\lim_{x \to 2n^+} f(x) = \lim_{x \to 4m^+} f(x) = 8m = f(4m) = f(2n),$$

but

$$\lim_{x \to 2n^-} f(x) = \lim_{x \to 4m^-} f(x) = 8m - 6 \ne f(4m) = f(2n);$$

while if $n = 2m + 1$ for some integer m, then

$$\lim_{x \to 2n^+} f(x) = \lim_{x \to (4m+2)^+} f(x) = 8m + 2 = f(4m + 2) = f(2n),$$

but

$$\lim_{x \to 2n^-} f(x) = \lim_{x \to (4m+2)^-} f(x) = 8m \neq f(4m + 2) = f(2n).$$

Thus, f is right-continuous but not left-continuous at each discontinuity.

In Exercises 39–52, determine the domain of the function and prove that it is continuous on its domain using the Laws of Continuity and the facts quoted in this section.

39. $f(x) = 2 \sin x + 3 \cos x$

SOLUTION The domain of $2 \sin x + 3 \cos x$ is all real numbers. Because the trigonometric functions $\sin x$ and $\cos x$ are both continuous by Theorem 3, so are the functions $2 \sin x$ and $3 \cos x$ by Theorem 1(ii) and the function $2 \sin x + 3 \cos x$ by Theorem 1(i).

40. $f(x) = \sqrt{x^2 + 9}$

SOLUTION The domain of $\sqrt{x^2 + 9}$ is all real numbers, as $x^2 + 9 > 0$ for all x. Because $\sqrt{x}$ is continuous by Theorem 3 and the polynomial function $x^2 + 9$ is continuous by Theorem 2, so is the composite function $\sqrt{x^2 + 9}$ by Theorem 4.

41. $f(x) = \sqrt{x} \sin x$

SOLUTION The domain of $\sqrt{x} \sin x$ is $\{x | x \geq 0\}$. Because $\sqrt{x}$ and the trigonometric function $\sin x$ are both continuous by Theorem 3, so is $\sqrt{x} \sin x$ by Theorem 1(iii).

42. $f(x) = \dfrac{x^2}{x + x^{1/4}}$

SOLUTION This function is defined as long as $x \geq 0$ and $x + x^{1/4} \neq 0$, so the domain is all $\{x | x > 0\}$. On this domain, the polynomial functions x and x^2 are continuous by Theorem 2, and $x^{1/4}$ is continuous by Theorem 3. It follows that $x + x^{1/4}$ is continuous by Theorem 1(i). Finally, $\dfrac{x^2}{x + x^{1/4}}$ is continuous by Theorem 1(iv).

43. $f(x) = x^2 - 3x^{1/2}$

SOLUTION The domain of $x^2 - 3x^{1/2}$ is $\{x | x \geq 0\}$. On this domain, the polynomial function x^2 is continuous by Theorem 2, and $3x^{1/2}$ is continuous by Theorem 3. It follows that $x^2 - 3x^{1/2}$ is continuous by Theorem 1(i).

44. $f(x) = x^{1/3} + x^{3/4}$

SOLUTION The domain of $x^{1/3} + x^{3/4}$ is $\{x | x \geq 0\}$. On this domain, both $x^{1/3}$ and $x^{3/4}$ are continuous by Theorem 3, so $x^{1/3} + x^{3/4}$ is continuous by Theorem 1(i).

45. $f(x) = x^{-4/3}$

SOLUTION The domain of $x^{-4/3}$ is $\{x | x \neq 0\}$. Because the function $x^{4/3}$ is continuous by Theorem 3 and not equal to zero for $x \neq 0$, it follows that

$$x^{-4/3} = \frac{1}{x^{4/3}}$$

is continuous for $x \neq 0$ by Theorem 1(iv).

46. $f(x) = \cos^3 x$

SOLUTION The domain of $\cos x$ is all real x, so the domain of $\cos^3 x$ is also all real x. Because $\cos x$ is continuous on its domain by Theorem 3, by repeated application of Theorem 1(iii), it follows that $\cos^3 x$ is continuous as well.

47. $f(x) = \tan^2 x$

SOLUTION The domain of $\tan^2 x$ is all $\{x | x \neq \pm(2n - 1)\pi/2\}$ where n is a positive integer. Because $\tan x$ is continuous on this domain by Theorem 3 and Theorem 1(iv), it follows from Theorem 1(iii) that $\tan^2 x = \tan x \cdot \tan x$ is also continuous on this domain.

48. $f(x) = \cos(x^{1/3} + 1)$

SOLUTION The domain of $\cos(x^{1/3} + 1)$ is all real numbers. On this domain, $x^{1/3}$ is continuous by Theorem 3, and the polynomial 1 is continuous by Theorem 2. Therefore, $x^{1/3} + 1$ is continuous by Theorem 1(i). The function $\cos x$ is continuous by Theorem 3, so the composite function $\cos(x^{1/3} + 1)$ is continuous by Theorem 4.

49. $f(x) = (x^4 + 1)^{3/2}$

SOLUTION The domain of $(x^4 + 1)^{3/2}$ is all real numbers as $x^4 + 1 > 0$ for all x. Because $x^{3/2}$ is continuous by Theorem 3 and the polynomial function $x^4 + 1$ is continuous by Theorem 2, so is the composite function $(x^4 + 1)^{3/2}$ by Theorem 5.

50. $f(x) = (x^3 + 3)^{5/2}$

SOLUTION The domain of $(x^3 + 3)^{5/2}$ is $\{x | x > -\sqrt[3]{3}\}$. On this domain, the polynomial $x^3 + 3$ is continuous by Theorem 2, and is also non-negative. For non-negative inputs, $x^{5/2}$ is continuous by Theorem 3, so the composite function $(x^3 + 3)^{5/2}$ is continuous by Theorem 4.

51. $f(x) = \dfrac{\cos(x^2)}{x^2 - 1}$

SOLUTION The domain for this function is all $\{x | x \neq \pm 1\}$. Because the trigonometric function $\cos x$ and the polynomial function x^2 are continuous on this domain by Theorems 3 and 2, respectively, so is the composite function $\cos(x^2)$ by Theorem 5. Finally, because the polynomial function $x^2 - 1$ is continuous by Theorem 2 and not equal to zero for $x \neq \pm 1$, the function $\dfrac{\cos(x^2)}{x^2 - 1}$ is continuous by Theorem 1(iv).

52. $f(x) = \dfrac{\tan^3(x-2)}{9x^2 + 2}$

SOLUTION The domain of this function is $\{x | x \neq 2 \pm (2n - 1)\pi/2\}$, where n is a positive integer. The polynomial function $x - 2$ is continuous on this domain by Theorem 2, so $\tan^3(x - 2)$ is continuous on this domain by Theorems 1(iv), 3, and 4 and repeated application of Theorem 1(iii). Finally, because the polynomial $9x^2 + 2$ is continuous by Theorem 2 and never equal to 0, the function $\frac{\tan^3(x-2)}{9x^2+2}$ is continuous by Theorem 1(iv).

53. The graph of the following function is shown in Figure 18.

$$f(x) = \begin{cases} x^2 + 3 & \text{for } x < 1 \\ 10 - x & \text{for } 1 \le x \le 2 \\ 6x - x^2 & \text{for } x > 2 \end{cases}$$

Show that f is continuous for $x \neq 1, 2$. Then compute the right- and left-hand limits at $x = 1, 2$, and determine whether f is left-continuous, right-continuous, or continuous at these points.

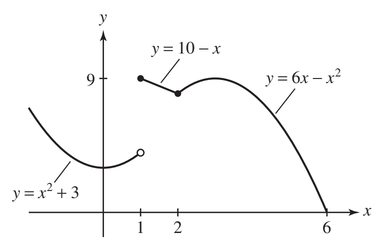


FIGURE 18

SOLUTION Let's start with $x \neq 1, 2$.

- The polynomial function $x^2 + 3$ is continuous by Theorem 2; therefore, $f(x)$ is continuous for $x < 1$.
- The polynomial function $10 - x$ is continuous by Theorem 2; therefore, $f(x)$ is continuous for $1 < x < 2$.
- The polynomial function $6x - x^2$ is continuous by Theorem 2; therefore, $f(x)$ is continuous for $x > 2$.

At $x = 1$, $f(x)$ has a jump discontinuity because the one-sided limits exist but are not equal:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2 + 3) = 4, \qquad \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (10 - x) = 9$$

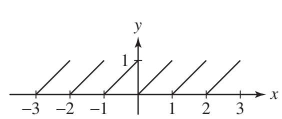
Furthermore, the right-hand limit equals the function value $f(1) = 9$, so $f(x)$ is right-continuous at $x = 1$. At $x = 2$,

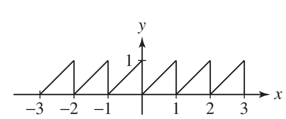
$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (10 - x) = 8, \qquad \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (6x - x^2) = 8$$

The left- and right-hand limits exist and are equal to $f(2)$, so $f(x)$ is continuous at $x = 2$.

54. Sawtooth Function Draw the graph of $f(x) = x - \lfloor x \rfloor$. At which points is f discontinuous? Is it left- or right-continuous at those points?

SOLUTION Two views of the sawtooth function $f(x) = x - \lfloor x \rfloor$ appear below. The first is the actual graph. In the second, the jumps are "connected" so as to better illustrate its "sawtooth" nature. The function is right-continuous at integer values of x.





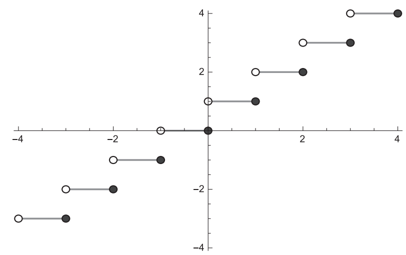
*In Exercises 55–56, $\lceil x \rceil$ refers to the **least integer function**. It is defined by $\lceil x \rceil = n$, where n is the unique integer such that $n - 1 < x \le n$. In each case, provide the graph of f, indicate the points of discontinuity and type of each (removable, jump, infinite, or none of these), and indicate whether f is left- or right-continuous.*

55. $f(x) = \lceil x \rceil$

SOLUTION The graph of $f(x) = \lceil x \rceil$ is shown in the figure below. From the graph we see that f has a jump discontinuity at $x = n$ for all integers n. Because

$$\lim_{x\to n^-} f(x) = n = f(n) \quad \text{but} \quad \lim_{x\to n^+} f(x) = n + 1 \ne f(n)$$

f is left-continuous at each discontinuity.

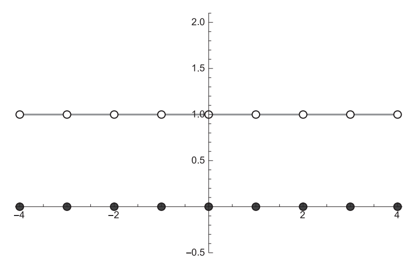


56. $f(x) = \lceil x \rceil - \lfloor x \rfloor$

SOLUTION If n is an integer, then $f(n) = \lceil n \rceil - \lfloor n \rfloor = n - n = 0$. If x is not an integer and n is the unique integer such that $n - 1 < x < n$, then $f(x) = \lceil x \rceil - \lfloor x \rfloor = n - (n - 1) = 1$. The graph of f is shown in the figure below. From the graph we see that f has a jump discontinuity at $x = n$ for all integers n. Because

$$\lim_{x\to n^-} f(x) = 1 \ne 0 = f(n) \quad \text{but} \quad \lim_{x\to n^+} f(x) = 1 \ne 0 = f(n)$$

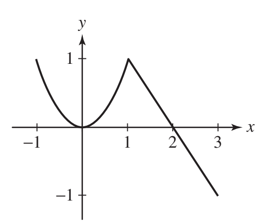
f is neither left- nor right-continuous at each discontinuity.



In Exercises 57–60, sketch the graph of f. At each point of discontinuity, state whether f is left- or right-continuous.

57. $f(x) = \begin{cases} x^2 & \text{for } x \le 1 \\ 2 - x & \text{for } x > 1 \end{cases}$

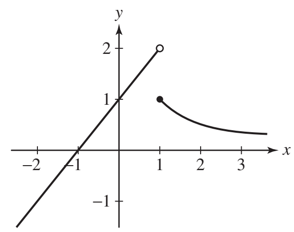
SOLUTION



The function f is continuous everywhere.

58. $f(x) = \begin{cases} x + 1 & \text{for } x < 1 \\ \dfrac{1}{x} & \text{for } x \ge 1 \end{cases}$

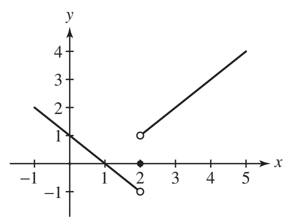
SOLUTION



The function f is right-continuous at $x = 1$.

59. $f(x) = \begin{cases} \dfrac{x^2 - 3x + 2}{|x - 2|} & x \neq 2 \\ 0 & x = 2 \end{cases}$

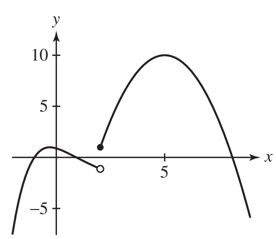
SOLUTION



The function is neither left-continuous nor right-continuous at $x = 2$.

60. $f(x) = \begin{cases} x^3 + 1 & \text{for } -\infty < x \leq 0 \\ -x + 1 & \text{for } 0 < x < 2 \\ -x^2 + 10x - 15 & \text{for } x \geq 2 \end{cases}$

SOLUTION



The function f is right-continuous at $x = 2$.

61. Show that the function

$$f(x) = \begin{cases} \dfrac{x^2 - 16}{x - 4} & x \neq 4 \\ 10 & x = 4 \end{cases}$$

has a removable discontinuity at $x = 4$.

SOLUTION Note that

$$\lim_{x \to 4} f(x) = \lim_{x \to 4} \frac{x^2 - 16}{x - 4} = \lim_{x \to 4} \frac{(x + 4)(x - 4)}{x - 4} = \lim_{x \to 4}(x + 4) = 8$$

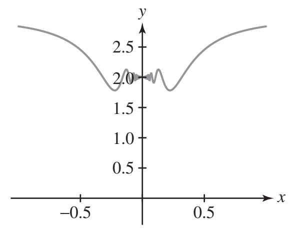
Because $\lim_{x \to 4} f(x)$ exists but is not equal to $f(4) = 10$, the function f has a removable discontinuity at $x = 4$.

62. GU Define $f(x) = x \sin \frac{1}{x} + 2$ for $x \neq 0$. Plot f. How should $f(0)$ be defined so that f is continuous at $x = 0$?

SOLUTION A plot of f is shown below. Based on this graph, it appears that

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \left(x \sin \frac{1}{x} + 2 \right) = 2$$

Therefore, $f(0)$ should be defined equal to 2 to make f continuous at $x = 0$.



In Exercises 63–64, H is the Heaviside function, defined by

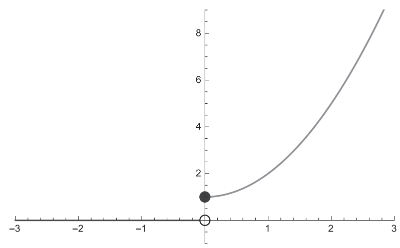
$$H(x) = \begin{cases} 0 & \text{when } x < 0 \\ 1 & \text{when } x \geq 0 \end{cases}$$

63. In each case, sketch the graph of f, indicate whether or not f is continuous, and—if f is not continuous—identify the points of discontinuity.

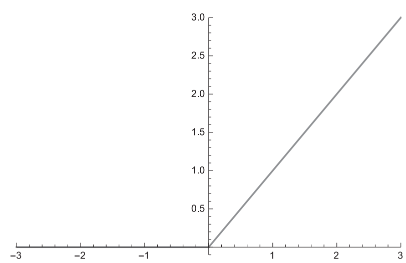
(a) $f(x) = H(x)(x^2 + 1)$
(b) $f(x) = H(x)x$
(c) $f(x) = H(x-2)\sqrt{x}$
(d) $f(x) = H(1+x)H(1-x)(1-x^2)$

SOLUTION

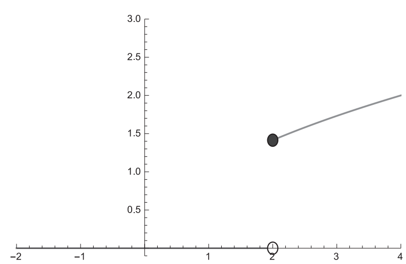
(a) The graph of $f(x) = H(x)(x^2 + 1)$ is shown below. From the graph, we see that f has a jump discontinuity at $x = 0$.



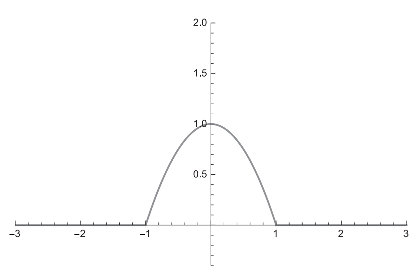
(b) The graph of $f(x) = H(x)x$ is shown below. From the graph, we see that f is continuous.



(c) The graph of $f(x) = H(x-2)\sqrt{x}$ is shown below. From the graph, we see that f has a jump discontinuity at $x = 2$.



(d) The graph of $f(x) = H(1+x)H(1-x)(1-x^2)$ is shown below. From the graph, we see that f is continuous.



64. Assume that f is defined and continuous for all x. Under what condition on f are we assured that the function g, defined by $g(x) = H(x-a)f(x)$, is continuous?

SOLUTION First note that $g(a) = H(0)f(a) = f(a)$. Now, for $x < a$, $H(x-a) = 0$ and $g(x) = 0$, while for $x > a$, $H(x-a) = 1$ and $g(x) = f(x)$. It follows that g is continuous for $x < a$ and for $x > a$. Moreover,

$$\lim_{x\to a^-} g(x) = 0 \quad \text{and} \quad \lim_{x\to a^+} g(x) = \lim_{x\to a^+} f(x) = f(a)$$

where this last result follows because f is continuous at $x = a$. For g to be continuous at $x = a$, $\lim_{x\to a} g(x)$ must exist and be equal to $g(a)$. This will only happen if $f(a) = 0$.

In Exercises 65–67, find the value of the constant (a, b, or c) that makes the function continuous.

65. $f(x) = \begin{cases} x^2 - c & \text{for } x < 5 \\ 4x + 2c & \text{for } x \ge 5 \end{cases}$

SOLUTION As $x \to 5^-$, we have $x^2 - c \to 25 - c = L$. As $x \to 5^+$, we have $4x + 2c \to 20 + 2c = R$. Match the limits: $L = R$ or $25 - c = 20 + 2c$ implies $c = \frac{5}{3}$.

66. $f(x) = \begin{cases} 2x + 9x^{-1} & \text{for } x \le 3 \\ -4x + c & \text{for } x > 3 \end{cases}$

SOLUTION As $x \to 3^-$, we have $2x + 9x^{-1} \to 9 = L$. As $x \to 3^+$, we have $-4x + c \to c - 12 = R$. Match the limits: $L = R$ or $9 = c - 12$ implies $c = 21$.

67. $f(x) = \begin{cases} x^{-1} & \text{for } x < -1 \\ ax + b & \text{for } -1 \le x \le \frac{1}{2} \\ x^{-1} & \text{for } x > \frac{1}{2} \end{cases}$

SOLUTION As $x \to -1^-$, we have $x^{-1} \to -1$, while as $x \to -1^+$, we have $ax + b \to -a + b$. Additionally, as $x \to \frac{1}{2}^-$, we have $ax + b \to \frac{1}{2}a + b$, while as $x \to \frac{1}{2}^+$, we have $x^{-1} \to 2$. In order for f to be continuous for all x, a and b must satisfy the system of equations

$$-1 = -a + b \qquad \text{and} \qquad \frac{1}{2}a + b = 2$$

The solution of this system of equations is $a = 2$ and $b = 1$.

68. Define

$$g(x) = \begin{cases} x + 3 & \text{for } x < -1 \\ cx & \text{for } -1 \le x \le 2 \\ x + 2 & \text{for } x > 2 \end{cases}$$

Find a value of c such that g is

(a) left-continuous

(b) right-continuous

In each case, sketch the graph of g.

SOLUTION

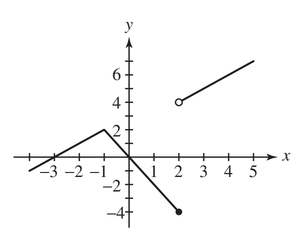
(a) In order for g to be left-continuous, we must have

$$\lim_{x\to -1^-} g(x) = g(-1)$$

Now,

$$\lim_{x\to -1^-} g(x) = \lim_{x\to -1^-} (x+3) = 2, \qquad \text{and} \qquad g(-1) = -c$$

Thus, for g to be left-continuous, we need $c = -2$. A graph of g with $c = -2$ is shown below.



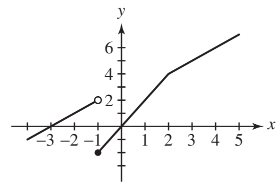
(b) In order for g to be right-continuous, we must have

$$\lim_{x\to 2^+} g(x) = g(2)$$

Now,

$$\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+} (x+2) = 4, \qquad \text{and} \qquad g(2) = 2c$$

Thus, for g to be right-continuous, we need $c = 2$. A graph of g with $c = 2$ is shown below.



69. Define $g(t) = \dfrac{t^3 - 1}{t^2 - 1}$ for $t \neq \pm 1$. Answer the following questions, using a plot if necessary.
(a) Can $g(1)$ be defined so that g is continuous at $t = 1$? If yes, how?
(b) Can $g(-1)$ be defined so that g is continuous at $t = -1$? If so, how?

SOLUTION
(a) Because

$$\lim_{t\to 1} \frac{t^3 - 1}{t^2 - 1} = \lim_{t\to 1} \frac{(t-1)(t^2+t+1)}{(t-1)(t+1)} = \lim_{t\to 1} \frac{t^2+t+1}{t+1} = \frac{3}{2}$$

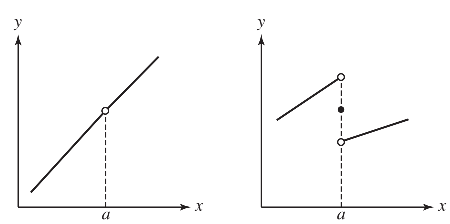
exists and is equal to $3/2$, defining $g(1) = 3/2$ will make g continuous at $t = 1$.
(b) Because the numerator of g approaches -2 while the denominator of g approaches 0 as $t \to -1$, it follows that

$$\lim_{t\to -1} \frac{t^3 - 1}{t^2 - 1}$$

does not exist. Therefore, it is not possible to define $g(-1)$ so that g is continuous at $t = -1$.

70. Each of the following statements is *false*. For each statement, sketch the graph of a function that provides a counterexample.
(a) If $\lim_{x\to a} f(x)$ exists, then f is continuous at $x = a$.
(b) If f has a jump discontinuity at $x = a$, then $f(a)$ is equal to either $\lim_{x\to a^-} f(x)$ or $\lim_{x\to a^+} f(x)$.

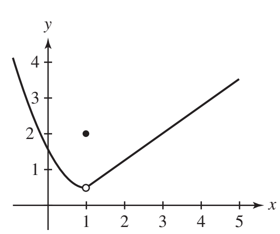
SOLUTION Refer to the two figures shown below.
(a) The figure at the left shows a function for which $\lim_{x\to a} f(x)$ exists, but the function is not continuous at $x = a$ because the function is not defined at $x = a$.
(b) The figure at the right shows a function that has a jump discontinuity at $x = a$ but $f(a)$ is not equal to either $\lim_{x\to a^-} f(x)$ or $\lim_{x\to a^-} f(x)$.



In Exercises 71–74, draw the graph of a function on $[0, 5]$ *with the given properties.*

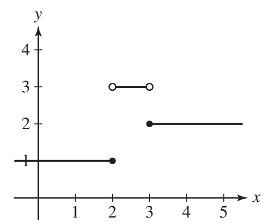
71. f is not continuous at $x = 1$, but $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 1^-} f(x)$ exist and are equal.

SOLUTION



72. f is left-continuous but not continuous at $x = 2$, and right-continuous but not continuous at $x = 3$.

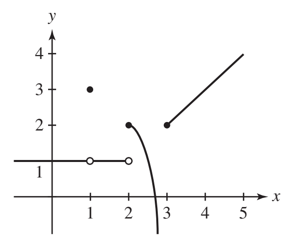
SOLUTION



73. f has a removable discontinuity at $x = 1$, a jump discontinuity at $x = 2$, and

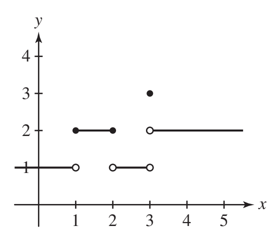
$$\lim_{x\to 3^-} f(x) = -\infty, \qquad \lim_{x\to 3^+} f(x) = 2$$

SOLUTION



74. f is right- but not left-continuous at $x = 1$, left- but not right-continuous at $x = 2$, and neither left- nor right-continuous at $x = 3$.

SOLUTION



In Exercises 75–86, evaluate using substitution.

75. $\lim_{x\to -1}(2x^3 - 4)$

SOLUTION $\lim_{x\to -1}(2x^3 - 4) = 2(-1)^3 - 4 = -6$

76. $\lim_{x\to 2}(5x - 12x^{-2})$

SOLUTION $\lim_{x\to 2}(5x - 12x^{-2}) = 5(2) - 12(2^{-2}) = 10 - 12(\frac{1}{4}) = 7$

77. $\lim_{x\to 3}\dfrac{x+2}{x^2+2x}$

SOLUTION $\lim_{x\to 3}\dfrac{x+2}{x^2+2x} = \dfrac{3+2}{3^2+2\cdot 3} = \dfrac{5}{15} = \dfrac{1}{3}$

78. $\lim_{x\to\pi}\sin\left(\dfrac{x}{2} - \pi\right)$

SOLUTION $\lim_{x\to\pi}\sin(\frac{x}{2} - \pi) = \sin(-\frac{\pi}{2}) = -1.$

79. $\lim_{x\to\frac{\pi}{4}}\tan(3x)$

SOLUTION $\lim_{x\to\frac{\pi}{4}}\tan(3x) = \tan(3\cdot\frac{\pi}{4}) = \tan(\frac{3\pi}{4}) = -1$

80. $\lim_{x\to\pi}\dfrac{1}{\cos x}$

SOLUTION $\lim_{x\to\pi} \frac{1}{\cos x} = \frac{1}{\cos \pi} = \frac{1}{-1} = -1$

81. $\lim_{x\to 4} x^{-5/2}$

SOLUTION $\lim_{x\to 4} x^{-5/2} = 4^{-5/2} = \frac{1}{32}$

82. $\lim_{x\to 2} \sqrt{x^3 + 4x}$

SOLUTION $\lim_{x\to 2} \sqrt{x^3 + 4x} = \sqrt{2^3 + 4(2)} = 4$

83. $\lim_{x\to -1} (1 - 8x^3)^{3/2}$

SOLUTION $\lim_{x\to -1} (1 - 8x^3)^{3/2} = (1 - 8(-1)^3)^{3/2} = 27$

84. $\lim_{x\to 2} \left(\frac{7x+2}{4-x}\right)^{2/3}$

SOLUTION $\lim_{x\to 2} \left(\frac{7x+2}{4-x}\right)^{2/3} = \left(\frac{7(2)+2}{4-2}\right)^{2/3} = 4$

85. $\lim_{x\to 3} 10^{x^2-2x}$

SOLUTION $\lim_{x\to 3} 10^{x^2-2x} = 10^{3^2-2(3)} = 1000$

86. $\lim_{x\to -\frac{\pi}{2}} 3^{\sin x}$

SOLUTION $\lim_{x\to -\frac{\pi}{2}} 3^{\sin x} = 3^{-\sin(\pi/2)} = \frac{1}{3}$

87. Suppose that f and g are discontinuous at $x = c$. Does it follow that $f + g$ is discontinuous at $x = c$? If not, give a counterexample. Does this contradict Theorem 1(i)?

SOLUTION Even if f and g are discontinuous at $x = c$, it is *not* necessarily true that $f + g$ is discontinuous at $x = c$. For example, suppose $f(x) = -x^{-1}$ and $g(x) = x^{-1}$. Both f and g are discontinuous at $x = 0$; however, the function $f(x) + g(x) = 0$ is continuous everywhere, including $x = 0$. This does not contradict Theorem 1 (i), which deals only with continuous functions.

88. Prove that $f(x) = |x|$ is continuous for all x. *Hint:* To prove continuity at $x = 0$, consider the one-sided limits.

SOLUTION Let $c > 0$. Then

$$\lim_{x\to c} f(x) = \lim_{x\to c} |x| = \lim_{x\to c} x = c = |c| = f(x)$$

and f is continuous at $x = c > 0$. Next, let $c < 0$. Then

$$\lim_{x\to c} f(x) = \lim_{x\to c} |x| = \lim_{x\to c} (-x) = -c = |c| = f(x)$$

and f is continuous at $x = c < 0$. Finally, let $c = 0$. Then

$$\lim_{x\to c^-} f(x) = \lim_{x\to 0^-} |x| = \lim_{x\to 0^-} (-x) = 0$$

and

$$\lim_{x\to c^+} f(x) = \lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0$$

Because the two one-sided limits are equal, it follows that

$$\lim_{x\to 0} f(x) = 0 = f(0)$$

and f is continuous at $x = c = 0$. Bringing these three pieces together, we see that $f(x) = |x|$ is continuous for all x.

89. Use the result of Exercise 88 to prove that if g is continuous, then $f(x) = |g(x)|$ is also continuous.

SOLUTION Let c be an arbitrary real number at which g is continuous. Following the logic of Exercise 88 depending upon whether $g(c)$ is positive, negative, or zero, we find

$$\lim_{x\to c} f(x) = \lim_{x\to c} |g(x)| = |g(c)| = f(c)$$

Thus, if g is continuous, then $f(x) = |g(x)|$ is continuous also.

90. Which of the following quantities would be represented by continuous functions of time and which would have one or more discontinuities?

(a) Velocity of an airplane during a flight
(b) Temperature in a room under ordinary conditions
(c) Value of a bank account with interest paid yearly
(d) Salary of a teacher
(e) Population of the world

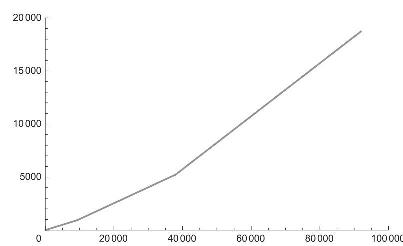
SOLUTION
(a) The velocity of an airplane during a flight is a continuous function of time.
(b) The temperature of a room under ordinary conditions is a continuous function of time.
(c) The value of a bank account with interest paid yearly is *not* a continuous function of time. It has discontinuities when deposits or withdrawals are made and when interest is paid.
(d) The salary of a teacher is *not* a continuous function of time. It has discontinuities whenever the teacher gets a raise (or whenever his or her salary is lowered).
(e) The population of the world is *not* a continuous function of time since it changes by a discrete amount with each birth or death. Since it takes on such large numbers (many billions), it is often treated as a continuous function for the purposes of mathematical modeling.

91. In 2017, the federal income tax T on income of x dollars (up to \$91,900) was determined by the formula

$$T(x) = \begin{cases} 0.10x & \text{for } 0 \le x < 9325 \\ 0.15x - 466.25 & \text{for } 9325 \le x < 37{,}950 \\ 0.25x - 4261.25 & \text{for } 37{,}950 \le x \le 91{,}900 \end{cases}$$

Sketch the graph of T. Does T have any discontinuities? Explain why, if T had a jump discontinuity, it might be advantageous in some situations to earn *less* money.

SOLUTION Here is a graph of $T(x)$ for 2017:



Note that the graph of T has no discontinuities. If $T(x)$ had a jump discontinuity (say at $x = c$), it might be advantageous to earn slightly less income than c (say $c - \epsilon$) and be taxed at a lower rate than to earn c or more and be taxed at a higher rate. Your net earnings may actually be more in the former case than in the latter one.

Further Insights and Challenges

92. If f has a removable discontinuity at $x = c$, then it is possible to redefine $f(c)$ so that f is continuous at $x = c$. Can this be done in more than one way?

SOLUTION In order for $f(x)$ to have a removable discontinuity at $x = c$, $\lim_{x\to c} f(x) = L$ must exist. To remove the discontinuity, we define $f(c) = L$. Then f is continuous at $x = c$ since $\lim_{x\to c} f(x) = L = f(c)$. Now *assume* that we may define $f(c) = M \neq L$ and still have f continuous at $x = c$. Then $\lim_{x\to c} f(x) = f(c) = M$. Therefore $M = L$, a contradiction. Roughly speaking, there's only one way to fill in the hole in the graph of f!

93. Give an example of functions f and g such that $f(g(x))$ is continuous but g has at least one discontinuity.

SOLUTION Answers may vary. The simplest examples are the functions $f(g(x))$ where $f(x) = C$ is a constant function, and $g(x)$ is defined for all x. In these cases, $f(g(x)) = C$. For example, if $f(x) = 3$ and $g(x) = [x]$, g is discontinuous at all integer values $x = n$, but $f(g(x)) = 3$ is continuous.

94. Continuous at Only One Point Show that the following function is continuous only at $x = 0$:

$$f(x) = \begin{cases} x & \text{for } x \text{ rational} \\ -x & \text{for } x \text{ irrational} \end{cases}$$

SOLUTION Let $f(x) = x$ for x rational and $f(x) = -x$ for x irrational.

- Now $f(0) = 0$ since 0 is rational. Moreover, as $x \to 0$, we have $|f(x) - f(0)| = |f(x) - 0| = |x| \to 0$. Thus $\lim_{x\to 0} f(x) = f(0)$ and f is continuous at $x = 0$.

- Let $c \neq 0$ be any nonzero rational number. Let $\{x_1, x_2, \ldots\}$ be a sequence of irrational points that approach c; that is, as $n \to \infty$, the x_n get arbitrarily close to c. Notice that as $n \to \infty$, we have $|f(x_n) - f(c)| = |-x_n - c| = |x_n + c| \to |2c| \neq 0$. Therefore, it is *not* true that $\lim_{x\to c} f(x) = f(c)$. Accordingly, f is *not* continuous at $x = c$. Since c was arbitrary, f is discontinuous at all rational numbers.
- Let $c \neq 0$ be any nonzero irrational number. Let $\{x_1, x_2, \ldots\}$ be a sequence of rational points that approach c; that is, as $n \to \infty$, the x_n get arbitrarily close to c. Notice that as $n \to \infty$, we have $|f(x_n) - f(c)| = |x_n - (-c)| = |x_n + c| \to |2c| \neq 0$. Therefore, it is *not* true that $\lim_{x\to c} f(x) = f(c)$. Accordingly, f is *not* continuous at $x = c$. Since c was arbitrary, f is discontinuous at all irrational numbers.
- CONCLUSION: f is continuous at $x = 0$ and is discontinuous at all points $x \neq 0$.

95. Show that f is a discontinuous function for all x, where $f(x)$ is defined as follows:

$$f(x) = \begin{cases} 1 & \text{for } x \text{ rational} \\ -1 & \text{for } x \text{ irrational} \end{cases}$$

Show that f^2 is continuous for all x.

SOLUTION $\lim_{x\to c} f(x)$ does not exist for any c. If c is irrational, then there is always a rational number r arbitrarily close to c so that $|f(c) - f(r)| = 2$. If, on the other hand, c is rational, there is always an *irrational* number z arbitrarily close to c so that $|f(c) - f(z)| = 2$.

On the other hand, $f(x)^2$ is a constant function that always has value 1, which is obviously continuous.

2.5 Indeterminate Forms

Preliminary Questions

1. Which of the following is indeterminate at $x = 1$?

$$\frac{x^2+1}{x-1}, \qquad \frac{x^2-1}{x+2}, \qquad \frac{x^2-1}{\sqrt{x+3}-2}, \qquad \frac{x^2+1}{\sqrt{x+3}-2}$$

SOLUTION At $x = 1$, $\frac{x^2-1}{\sqrt{x+3}-2}$ is of the form $\frac{0}{0}$; hence, this function is indeterminate. None of the remaining functions is indeterminate at $x = 1$: $\frac{x^2+1}{x-1}$ and $\frac{x^2+1}{\sqrt{x+3}-2}$ are undefined because the denominator is zero but the numerator is not, while $\frac{x^2-1}{x+2}$ is equal to 0.

2. Give counterexamples to show that these statements are false:

(a) If $f(c)$ is indeterminate, then the right- and left-hand limits as $x \to c$ are not equal.

(b) If $\lim_{x\to c} f(x)$ exists, then $f(c)$ is not indeterminate.

(c) If $f(x)$ is undefined at $x = c$, then $f(x)$ has an indeterminate form at $x = c$.

SOLUTION

(a) Let $f(x) = \frac{x^2-1}{x-1}$. At $x = 1$, f is indeterminate of the form $\frac{0}{0}$ but

$$\lim_{x\to 1^-} \frac{x^2-1}{x-1} = \lim_{x\to 1^-} (x+1) = 2 = \lim_{x\to 1^+} (x+1) = \lim_{x\to 1^+} \frac{x^2-1}{x-1}$$

(b) Again, let $f(x) = \frac{x^2-1}{x-1}$. Then

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1}(x+1) = 2$$

but $f(1)$ is indeterminate of the form $\frac{0}{0}$.

(c) Let $f(x) = \frac{1}{x}$. Then f is undefined at $x = 0$ but does not have an indeterminate form at $x = 0$.

3. The method for evaluating limits discussed in this section is sometimes called simplify and plug in. Explain how it actually relies on the property of continuity.

SOLUTION If f is continuous at $x = c$, then, by definition, $\lim_{x\to c} f(x) = f(c)$; in other words, the limit of a continuous function at $x = c$ is the value of the function at $x = c$. The "simplify and plug-in" strategy is based on simplifying a function which is indeterminate to a continuous function. Once the simplification has been made, the limit of the remaining continuous function is obtained by evaluation.

Exercises

In Exercises 1–4, show that the limit leads to an indeterminate form. Then carry out the two-step procedure: Transform the function algebraically and evaluate using continuity.

1. $\lim_{x\to 6} \frac{x^2-36}{x-6}$

SOLUTION When we substitute $x = 6$ into $\frac{x^2-36}{x-6}$, we obtain the indeterminate form $\frac{0}{0}$. Upon factoring the numerator and denominator and then simplifying, we find

$$\lim_{x\to 6} \frac{x^2-36}{x-6} = \lim_{x\to 6} \frac{(x-6)(x+6)}{x-6} = \lim_{x\to 6}(x+6) = 12$$

2. $\lim_{h\to 3} \frac{9-h^2}{h-3}$

SOLUTION When we substitute $h = 3$ into $\frac{9-h^2}{h-3}$, we obtain the indeterminate form $\frac{0}{0}$. Upon factoring the numerator and denominator and then simplifying, we find

$$\lim_{h\to 3} \frac{9-h^2}{h-3} = \lim_{h\to 3} \frac{(3-h)(3+h)}{h-3} = \lim_{h\to 3} -(3+h) = -6$$

3. $\lim_{x\to -1} \frac{x^2+2x+1}{x+1}$

SOLUTION When we substitute $x = -1$ into $\frac{x^2+2x+1}{x+1}$, we obtain the indeterminate form $\frac{0}{0}$. Upon factoring the numerator and simplifying, we find

$$\lim_{x\to -1} \frac{x^2+x+1}{x+1} = \lim_{x\to -1} \frac{(x+1)^2}{x+1} = \lim_{x\to -1}(x+1) = 0$$

4. $\lim_{t\to 9} \frac{2t-18}{5t-45}$

SOLUTION When we substitute $t = 9$ into $\frac{2t-18}{5t-45}$, we obtain the indeterminate form $\frac{0}{0}$. Upon factoring the numerator and denominator and then simplifying, we find

$$\lim_{t\to 9} \frac{2t-18}{5t-45} = \lim_{t\to 9} \frac{2(t-9)}{5(t-9)} = \lim_{t\to 9} \frac{2}{5} = \frac{2}{5}$$

In Exercises 5–34, evaluate the limit, if it exists. If not, determine whether the one-sided limits exist (finite or infinite).

5. $\lim_{x\to 7} \frac{x-7}{x^2-49}$

SOLUTION $\lim_{x\to 7} \frac{x-7}{x^2-49} = \lim_{x\to 7} \frac{x-7}{(x-7)(x+7)} = \lim_{x\to 7} \frac{1}{x+7} = \frac{1}{14}$

6. $\lim_{x\to 8} \frac{x^2-64}{x-9}$

SOLUTION $\lim_{x\to 8} \frac{x^2-64}{x-9} = \frac{0}{-1} = 0$

7. $\lim_{x\to -2} \frac{x^2+3x+2}{x+2}$

SOLUTION $\lim_{x\to -2} \frac{x^2+3x+2}{x+2} = \lim_{x\to -2} \frac{(x+2)(x+1)}{x+2} = \lim_{x\to -2}(x+1) = -1$

8. $\lim_{x\to 8} \frac{x^3-64x}{x-8}$

SOLUTION $\lim_{x\to 8} \frac{x^3-64x}{x-8} = \lim_{x\to 8} \frac{x(x-8)(x+8)}{x-8} = \lim_{x\to 8} x(x+8) = 8(16) = 128$

9. $\lim_{x\to 5} \frac{2x^2-9x-5}{x^2-25}$

SOLUTION $\lim_{x\to 5} \frac{2x^2-9x-5}{x^2-25} = \lim_{x\to 5} \frac{(2x+1)(x-5)}{(x-5)(x+5)} = \lim_{x\to 5} \frac{2x+1}{x+5} = \frac{11}{10}.$

10. $\lim_{h\to 0} \frac{(1+h)^3-1}{h}$

SOLUTION

$$\lim_{h\to 0}\frac{(1+h)^3-1}{h}=\lim_{h\to 0}\frac{1+3h+3h^2+h^3-1}{h}=\lim_{h\to 0}\frac{3h+3h^2+h^3}{h}$$
$$=\lim_{h\to 0}(3+3h+h^2)=3+3(0)+0^2=3$$

11. $\lim\limits_{x\to -\frac{1}{2}}\frac{2x+1}{2x^2+3x+1}$

SOLUTION $\lim\limits_{x\to -\frac{1}{2}}\frac{2x+1}{2x^2+3x+1}=\lim\limits_{x\to -\frac{1}{2}}\frac{2x+1}{(2x+1)(x+1)}=\lim\limits_{x\to -\frac{1}{2}}\frac{1}{x+1}=\frac{1}{1/2}=2$

12. $\lim\limits_{x\to 3}\frac{x^2-x}{x^2-9}$

SOLUTION Observe that as $x\to 3$, $x^2-x\to 6\neq 0$ and $x^2-9\to 0$. Accordingly,

$$\lim_{x\to 3}\frac{x^2-x}{x^2-9}\text{ does not exist.}$$

As for the one-sided limits, $x^2-x\to 6$ and $x^2-9\to 0^-$ as $x\to 3^-$; therefore,

$$\lim_{x\to 3^-}\frac{x^2-x}{x^2-9}=-\infty$$

On the other hand, $x^2-x\to 6$ and $x^2-9\to 0^+$ as $x\to 3^+$; therefore,

$$\lim_{x\to 3^+}\frac{x^2-x}{x^2-9}=\infty$$

13. $\lim\limits_{x\to 2}\frac{3x^2-4x-4}{2x^2-8}$

SOLUTION $\lim\limits_{x\to 2}\frac{3x^2-4x-4}{2x^2-8}=\lim\limits_{x\to 2}\frac{(3x+2)(x-2)}{2(x-2)(x+2)}=\lim\limits_{x\to 2}\frac{3x+2}{2(x+2)}=\frac{8}{8}=1$

14. $\lim\limits_{h\to 0}\frac{(3+h)^3-27}{h}$

SOLUTION

$$\lim_{h\to 0}\frac{(3+h)^3-27}{h}=\lim_{h\to 0}\frac{27+27h+9h^2+h^3-27}{h}=\lim_{h\to 0}\frac{27h+9h^2+h^3}{h}$$
$$=\lim_{h\to 0}(27+9h+h^2)=27$$

15. $\lim\limits_{t\to 0}\frac{4^{2t}-1}{4^t-1}$

SOLUTION $\lim\limits_{t\to 0}\frac{4^{2t}-1}{4^t-1}=\lim\limits_{t\to 0}\frac{(4^t-1)(4^t+1)}{4^t-1}=\lim\limits_{t\to 0}(4^t+1)=4^0+1=2$

16. $\lim\limits_{h\to 4}\frac{(h+2)^2-9h}{h-4}$

SOLUTION $\lim\limits_{h\to 4}\frac{(h+2)^2-9h}{h-4}=\lim\limits_{h\to 4}\frac{h^2-5h+4}{h-4}=\lim\limits_{h\to 4}\frac{(h-1)(h-4)}{h-4}=\lim\limits_{h\to 4}(h-1)=3$

17. $\lim\limits_{x\to 16}\frac{\sqrt{x}-4}{x-16}$

SOLUTION $\lim\limits_{x\to 16}\frac{\sqrt{x}-4}{x-16}=\lim\limits_{x\to 16}\frac{\sqrt{x}-4}{\left(\sqrt{x}+4\right)\left(\sqrt{x}-4\right)}=\lim\limits_{x\to 16}\frac{1}{\sqrt{x}+4}=\frac{1}{8}$

18. $\lim\limits_{t\to -2}\frac{2t+4}{12-3t^2}$

SOLUTION $\lim\limits_{t\to -2}\frac{2t+4}{12-3t^2}=\lim\limits_{t\to -2}\frac{2(t+2)}{-3(t-2)(t+2)}=\lim\limits_{t\to -2}\frac{2}{-3(t-2)}=\frac{1}{6}.$

19. $\lim\limits_{h\to 0}\frac{\frac{1}{(h+2)^2}-\frac{1}{4}}{h}$

SOLUTION

$$\lim_{h\to 0}\frac{\frac{1}{(h+2)^2}-\frac{1}{4}}{h}=\lim_{h\to 0}\frac{\frac{4-(h+2)^2}{4(h+2)^2}}{h}=\lim_{h\to 0}\frac{\frac{4-(h^2+4h+4)}{4(h+2)^2}}{h}=\lim_{h\to 0}\frac{\frac{-h^2-4h}{4(h+2)^2}}{h}$$
$$=\lim_{h\to 0}\frac{h\frac{-h-4}{4(h+2)^2}}{h}=\lim_{h\to 0}\frac{-h-4}{4(h+2)^2}=\frac{-4}{16}=-\frac{1}{4}$$

20. $\lim_{y\to 3}\frac{y^2+y-12}{y^3-10y+3}$

SOLUTION $\lim_{y\to 3}\frac{y^2+y-12}{y^3-10y+3}=\lim_{y\to 3}\frac{(y-3)(y+4)}{(y-3)(y^2+3y-1)}\lim_{y\to 3}\frac{y+4}{y^2+3y-1}=\frac{7}{17}$

21. $\lim_{h\to 0}\frac{\sqrt{2+h}-2}{h}$

SOLUTION Observe that as $h\to 0$, $\sqrt{2+h}-2\to\sqrt{2}-2\neq 0$ and $h\to 0$. Accordingly,

$$\lim_{h\to 0}\frac{\sqrt{2+h}-2}{h}\text{ does not exist.}$$

As for the one-sided limits, $\sqrt{2+h}-2\to\sqrt{2}-2<0$ and $h\to 0^-$ as $h\to 0^-$; therefore,

$$\lim_{h\to 0^-}\frac{\sqrt{2+h}-2}{h}=\infty$$

On the other hand, $\sqrt{2+h}-2\to\sqrt{2}-2<0$ and $h\to 0^+$ as $h\to 0^+$; therefore,

$$\lim_{h\to 0^+}\frac{\sqrt{2+h}-2}{h}=-\infty$$

22. $\lim_{x\to 8}\frac{\sqrt{x-4}-2}{x-8}$

SOLUTION

$$\lim_{x\to 8}\frac{\sqrt{x-4}-2}{x-8}=\lim_{x\to 8}\left(\frac{\sqrt{x-4}-2}{x-8}\cdot\frac{\sqrt{x-4}+2}{\sqrt{x-4}+2}\right)=\lim_{x\to 8}\frac{x-8}{(x-8)(\sqrt{x-4}+2)}$$
$$=\lim_{x\to 8}\frac{1}{\sqrt{x-4}+2}=\frac{1}{\sqrt{4}+2}=\frac{1}{4}$$

23. $\lim_{x\to 4}\frac{x-4}{\sqrt{x}-\sqrt{8-x}}$

SOLUTION

$$\lim_{x\to 4}\frac{x-4}{\sqrt{x}-\sqrt{8-x}}=\lim_{x\to 4}\left(\frac{x-4}{\sqrt{x}-\sqrt{8-x}}\cdot\frac{\sqrt{x}+\sqrt{8-x}}{\sqrt{x}+\sqrt{8-x}}\right)=\lim_{x\to 4}\frac{(x-4)(\sqrt{x}+\sqrt{8-x})}{2x-8}$$
$$=\lim_{x\to 4}\frac{\sqrt{x}+\sqrt{8-x}}{2}=\frac{\sqrt{4}+\sqrt{4}}{2}=2$$

24. $\lim_{x\to 4}\frac{\sqrt{5-x}-1}{2-\sqrt{x}}$

SOLUTION

$$\lim_{x\to 4}\frac{\sqrt{5-x}-1}{2-\sqrt{x}}=\lim_{x\to 4}\left(\frac{\sqrt{5-x}-1}{2-\sqrt{x}}\cdot\frac{\sqrt{5-x}+1}{\sqrt{5-x}+1}\right)=\lim_{x\to 4}\frac{4-x}{(2-\sqrt{x})(\sqrt{5-x}+1)}$$
$$=\lim_{x\to 4}\frac{(2-\sqrt{x})(2+\sqrt{x})}{(2-\sqrt{x})(\sqrt{5-x}+1)}=\lim_{x\to 4}\frac{2+\sqrt{x}}{\sqrt{5-x}+1}=2$$

25. $\lim_{x\to 4}\left(\frac{1}{\sqrt{x}-2}-\frac{4}{x-4}\right)$

SOLUTION $\lim_{x\to 4}\left(\frac{1}{\sqrt{x}-2}-\frac{4}{x-4}\right)=\lim_{x\to 4}\frac{\sqrt{x}+2-4}{\left(\sqrt{x}-2\right)\left(\sqrt{x}+2\right)}=\lim_{x\to 4}\frac{\sqrt{x}-2}{\left(\sqrt{x}-2\right)\left(\sqrt{x}+2\right)}=\frac{1}{4}$

26. $\lim_{x\to 0^+}\left(\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{x^2+x}}\right)$

SOLUTION

$$\lim_{x\to 0^+}\left(\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{x^2+x}}\right)=\lim_{x\to 0^+}\frac{\sqrt{x+1}-1}{\sqrt{x}\sqrt{x+1}}=\lim_{x\to 0^+}\frac{\left(\sqrt{x+1}-1\right)\left(\sqrt{x+1}+1\right)}{\sqrt{x}\sqrt{x+1}\left(\sqrt{x+1}+1\right)}$$

$$=\lim_{x\to 0^+}\frac{x}{\sqrt{x}\sqrt{x+1}\left(\sqrt{x+1}+1\right)}=\lim_{x\to 0^+}\frac{\sqrt{x}}{\sqrt{x+1}\left(\sqrt{x+1}+1\right)}=0$$

27. $\lim_{x\to 0}\frac{\cot x}{\csc x}$

SOLUTION $\lim_{x\to 0}\frac{\cot x}{\csc x}=\lim_{x\to 0}\frac{\cos x}{\sin x}\cdot\sin x=\cos 0=1$

28. $\lim_{\theta\to\frac{\pi}{2}}\frac{\cot\theta}{\csc\theta}$

SOLUTION $\lim_{\theta\to\frac{\pi}{2}}\frac{\cot\theta}{\csc\theta}=\lim_{\theta\to\frac{\pi}{2}}\frac{\cos\theta}{\sin\theta}\cdot\sin\theta=\cos\frac{\pi}{2}=0$

29. $\lim_{x\to 1}\left(\frac{1}{1-x}-\frac{2}{1-x^2}\right)$

SOLUTION $\lim_{t\to 1}\left(\frac{1}{1-x}-\frac{2}{1-x^2}\right)=\lim_{t\to 1}\frac{x-1}{1-x^2}=\lim_{x\to 1}\frac{x-1}{(1-x)(1+x)}=\lim_{x\to 1}\frac{-1}{1+x}=-\frac{1}{2}$

30. $\lim_{x\to\frac{\pi}{4}}\frac{\sin x-\cos x}{\tan x-1}$

SOLUTION $\lim_{x\to\frac{\pi}{4}}\frac{\sin x-\cos x}{\tan x-1}\cdot\frac{\cos x}{\cos x}=\lim_{x\to\frac{\pi}{4}}\frac{(\sin x-\cos x)\cos x}{\sin x-\cos x}=\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2}$

31. $\lim_{t\to 2}\frac{2^{2t}+2^t-20}{2^t-4}$

SOLUTION $\lim_{t\to 2}\frac{2^{2t}+2^t-20}{2^t-4}=\lim_{t\to 2}\frac{(2^t+5)(2^t-4)}{2^t-4}=\lim_{t\to 2}(2^t+5)=9$

32. $\lim_{\theta\to\frac{\pi}{2}}(\sec\theta-\tan\theta)$

SOLUTION

$$\lim_{\theta\to\frac{\pi}{2}}(\sec\theta-\tan\theta)=\lim_{\theta\to\frac{\pi}{2}}\frac{1-\sin\theta}{\cos\theta}\cdot\frac{1+\sin\theta}{1+\sin\theta}=\lim_{\theta\to\frac{\pi}{2}}\frac{1-\sin^2\theta}{\cos\theta(1+\sin\theta)}=\lim_{\theta\to\frac{\pi}{2}}\frac{\cos\theta}{1+\sin\theta}=\frac{0}{2}=0$$

33. $\lim_{\theta\to\frac{\pi}{4}}\left(\frac{1}{\tan\theta-1}-\frac{2}{\tan^2\theta-1}\right)$

SOLUTION $\lim_{\theta\to\frac{\pi}{4}}\left(\frac{1}{\tan\theta-1}-\frac{2}{\tan^2\theta-1}\right)=\lim_{\theta\to\frac{\pi}{4}}\frac{\tan\theta-1}{\tan^2\theta-1}=\lim_{\theta\to\frac{\pi}{4}}\frac{\tan\theta-1}{(\tan\theta-1)(\tan\theta+1)}=\lim_{\theta\to\frac{\pi}{4}}\frac{1}{\tan\theta+1}=\frac{1}{1+1}=\frac{1}{2}$

34. $\lim_{x\to\frac{\pi}{3}}\frac{2\cos^2 x+3\cos x-2}{2\cos x-1}$

SOLUTION

$$\lim_{x\to\frac{\pi}{3}}\frac{2\cos^2 x+3\cos x-2}{2\cos x-1}=\lim_{x\to\frac{\pi}{3}}\frac{(2\cos x-1)(\cos x+2)}{2\cos x-1}=\lim_{x\to\frac{\pi}{3}}\cos x+2=\cos\frac{\pi}{3}+2=\frac{5}{2}$$

35. The following limits all have the indeterminate form 0/0. One of the limits does not exist, one is equal to 0, and one is a nonzero limit. Evaluate each limit algebraically if you can or investigate it numerically if you cannot.

- $\lim_{x\to -2}\frac{x^2+3x+2}{x+2}$

- $\lim_{x \to 1} \frac{1 - x^{-1}}{x - 2 + x^{-1}}$
- $\lim_{x \to 0} \frac{x^2}{1 - 5^x}$

SOLUTION

- $\lim_{x \to -2} \frac{x^2 + 3x + 2}{x + 2} = \lim_{x \to -2} \frac{(x+2)(x+1)}{x+2} = \lim_{x \to -2} (x+1) = -1$
- $\lim_{x \to 1} \frac{1 - x^{-1}}{x - 2 + x^{-1}} = \lim_{x \to 1} \frac{x - 1}{x^2 - 2x + 1} = \lim_{x \to 1} \frac{x-1}{(x-1)^2} = \lim_{x \to 1} \frac{1}{x-1}$, which does not exist because the numerator approaches $1 \neq 0$ while the denominator approaches 0.
- We will investigate this limit numerically. From the table below, it appears that $\lim_{x \to 0} \frac{x^2}{1 - 5^x} = 0$.

x	−0.01	−0.001	−0.0001	0.0001	0.001	0.01
$\frac{x^2}{1-5^x}$	0.00626348	0.00062184	0.00006214	−0.00006213	−0.00062084	−0.00616348

36. The following limits all have the indeterminate form ∞/∞. One of the limits does not exist, one is equal to 0, and one is a nonzero limit. Evaluate each limit algebraically if you can or investigate it numerically if you cannot.

- $\lim_{x \to 0} \frac{x^{-4}}{4 + x^{-1}}$
- $\lim_{x \to 0} \frac{3 \cot x}{\csc x}$
- $\lim_{x \to 0} \frac{1 + \frac{1}{x^2}}{1 + \frac{1}{x^4}}$

SOLUTION

- $\lim_{x \to 0} \frac{x^{-4}}{4 + x^{-1}} = \lim_{x \to 0} \frac{1}{4x^4 + x^3}$, which does not exist because the numerator approaches $1 \neq 0$ while the denominator approaches 0.
- $\lim_{x \to 0} \frac{3 \cot x}{\csc x} = \lim_{x \to 0} \frac{3 \cos x / \sin x}{1/\sin x} = \lim_{x \to 0} 3 \cos x = 3$
- $\lim_{x \to 0} \frac{1 + \frac{1}{x^2}}{1 + \frac{1}{x^4}} = \lim_{x \to 0} \frac{x^4 + x^2}{x^4 + 1} = 0$

In Exercises 37 and 38, show that the limit is in an indeterminate form, then investigate the limit numerically to estimate the value.

37. $\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta^2}$

SOLUTION When we substitute $\theta = 0$ into $\frac{1-\cos\theta}{\theta^2}$, we obtain the indeterminate form $\frac{0}{0}$. From the values in the table below, it appears that

$$\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta^2} = \frac{1}{2}.$$

θ	−0.1	−0.01	−0.001	0.001	0.01	0.1
$\frac{1 - \cos\theta}{\theta^2}$	0.49958347	0.49999583	0.49999996	0.49999996	0.49999583	0.49958347

38. $\lim_{\theta \to 0} \frac{1 - \cos^2\theta}{\theta^2}$

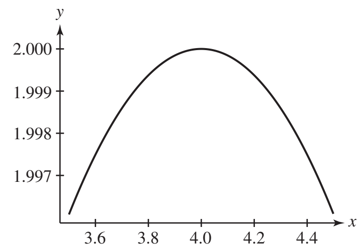
SOLUTION When we substitute $\theta = 0$ into $\frac{1-\cos^2\theta}{\theta^2}$, we obtain the indeterminate form $\frac{0}{0}$. From the values in the table below, it appears that

$$\lim_{\theta \to 0} \frac{1 - \cos^2\theta}{\theta^2} = 1.$$

θ	−0.1	−0.01	−0.001	0.001	0.01	0.1
$\frac{1 - \cos^2\theta}{\theta^2}$	0.99667111	0.99996667	0.99999967	0.99999967	0.99996667	0.99667111

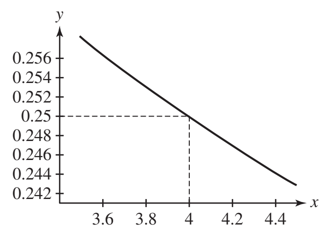
39. GU Use a plot of $f(x) = \dfrac{x-4}{\sqrt{x} - \sqrt{8-x}}$ to estimate $\lim_{x\to 4} f(x)$ to two decimal places. Compare with the answer obtained algebraically in Exercise 23.

SOLUTION Let $f(x) = \frac{x-4}{\sqrt{x}-\sqrt{8-x}}$. From the plot of $f(x)$ shown below, we estimate $\lim_{x\to 4} f(x) \approx 2.00$; to two decimal places, this matches the value of 2 obtained in Exercise 23.



40. GU Use a plot of $f(x) = \dfrac{1}{\sqrt{x}-2} - \dfrac{4}{x-4}$ to estimate $\lim_{x\to 4} f(x)$ numerically. Compare with the answer obtained algebraically in Exercise 25.

SOLUTION Let $f(x) = \frac{1}{\sqrt{x}-2} - \frac{4}{x-4}$. From the plot of $f(x)$ shown below, we estimate $\lim_{x\to 4} f(x) \approx 0.25$; to two decimal places, this matches the value of $\frac{1}{4}$ obtained in Exercise 25.



In Exercises 41–46, evaluate using the identity

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

41. $\lim_{x\to 2} \dfrac{x^3 - 8}{x - 2}$

SOLUTION $\lim_{x\to 2} \dfrac{x^3-8}{x-2} = \lim_{x\to 2} \dfrac{(x-2)(x^2+2x+4)}{x-2} = \lim_{x\to 2}(x^2+2x+4) = 12$

42. $\lim_{x\to 3} \dfrac{x^3 - 27}{x^2 - 9}$

SOLUTION $\lim_{x\to 3} \dfrac{x^3-27}{x^2-9} = \lim_{x\to 3} \dfrac{(x-3)(x^2+3x+9)}{(x-3)(x+3)} = \lim_{x\to 3} \dfrac{x^2+3x+9}{x+3} = \dfrac{27}{6} = \dfrac{9}{2}$

43. $\lim_{x\to 1} \dfrac{x^2 - 5x + 4}{x^3 - 1}$

SOLUTION $\lim_{x\to 1} \dfrac{x^2-5x+4}{x^3-1} = \lim_{x\to 1} \dfrac{(x-1)(x-4)}{(x-1)(x^2+x+1)} = \lim_{x\to 1} \dfrac{x-4}{x^2+x+1} = \dfrac{-3}{3} = -1$

44. $\lim_{x\to -2} \dfrac{x^3 + 8}{x^2 + 6x + 8}$

SOLUTION $\lim_{x\to -2} \dfrac{x^3+8}{x^2+6x+8} = \lim_{x\to -2} \dfrac{(x+2)(x^2-2x+4)}{(x+2)(x+4)} = \lim_{x\to -2} \dfrac{x^2-2x+4}{x+4} = \dfrac{12}{2} = 6$

45. $\lim_{x\to 1} \dfrac{x^4 - 1}{x^3 - 1}$

SOLUTION

$$\lim_{x\to 1} \frac{x^4-1}{x^3-1} = \lim_{x\to 1} \frac{(x^2-1)(x^2+1)}{(x-1)(x^2+x+1)} = \lim_{x\to 1} \frac{(x-1)(x+1)(x^2+1)}{(x-1)(x^2+x+1)} = \lim_{x\to 1} \frac{(x+1)(x^2+1)}{x^2+x+1} = \frac{4}{3}$$

46. $\lim_{x\to 27} \dfrac{x - 27}{x^{1/3} - 3}$

SOLUTION $\lim_{x\to 27}\frac{x-27}{x^{1/3}-3}=\lim_{x\to 27}\frac{(x^{1/3}-3)(x^{2/3}+3x^{1/3}+9)}{x^{1/3}-3}=\lim_{x\to 27}(x^{2/3}+3x^{1/3}+9)=27$

In Exercises 47–54, evaluate in terms of the constant a.

47. $\lim_{x\to 0}(2a+x)$

SOLUTION $\lim_{x\to 0}(2a+x)=2a$

48. $\lim_{h\to -2}(4ah+7a)$

SOLUTION $\lim_{h\to -2}(4ah+7a)=-a$

49. $\lim_{t\to -1}(4t-2at+3a)$

SOLUTION $\lim_{t\to -1}(4t-2at+3a)=-4+5a$

50. $\lim_{x\to a}\frac{(x+a)^2-4x^2}{x-a}$

SOLUTION

$$\lim_{x\to a}\frac{(x+a)^2-4x^2}{x-a}=\lim_{x\to a}\frac{(x^2+2ax+a^2)-4x^2}{x-a}=\lim_{x\to a}\frac{-3x^2+2ax+a^2}{x-a}$$
$$=\lim_{x\to a}\frac{(a-x)(a+3x)}{x-a}=\lim_{x\to a}(-(a+3x))=-4a$$

51. $\lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}}{x-a}$

SOLUTION $\lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}}{x-a}=\lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}}{\left(\sqrt{x}-\sqrt{a}\right)\left(\sqrt{x}+\sqrt{a}\right)}=\lim_{x\to a}\frac{1}{\sqrt{x}+\sqrt{a}}=\frac{1}{2\sqrt{a}}$

52. $\lim_{h\to 0}\frac{\sqrt{a+2h}-\sqrt{a}}{h}$

SOLUTION

$$\lim_{h\to 0}\frac{\sqrt{a+2h}-\sqrt{a}}{h}=\lim_{h\to 0}\frac{\left(\sqrt{a+2h}-\sqrt{a}\right)\left(\sqrt{a+2h}+\sqrt{a}\right)}{h\left(\sqrt{a+2h}+\sqrt{a}\right)}$$
$$=\lim_{h\to 0}\frac{2h}{h\left(\sqrt{a+2h}+\sqrt{a}\right)}=\lim_{h\to 0}\frac{2}{\sqrt{a+2h}+\sqrt{a}}=\frac{1}{\sqrt{a}}$$

53. $\lim_{x\to 0}\frac{(x+a)^3-a^3}{x}$

SOLUTION $\lim_{x\to 0}\frac{(x+a)^3-a^3}{x}=\lim_{x\to 0}\frac{x^3+3x^2a+3xa^2+a^3-a^3}{x}=\lim_{x\to 0}(x^2+3xa+3a^2)=3a^2$

54. $\lim_{h\to a}\frac{\frac{1}{h}-\frac{1}{a}}{h-a}$

SOLUTION $\lim_{h\to a}\frac{\frac{1}{h}-\frac{1}{a}}{h-a}=\lim_{h\to a}\frac{\frac{a-h}{ah}}{h-a}=\lim_{h\to a}\frac{a-h}{ah}\frac{1}{h-a}=\lim_{h\to a}\frac{-1}{ah}=-\frac{1}{a^2}$

55. Evaluate $\lim_{h\to 0}\frac{\sqrt[4]{1+h}-1}{h}$. *Hint:* Set $x=\sqrt[4]{1+h}$, express h as a function of x, and rewrite as a limit as $x\to 1$.

SOLUTION Let $x=\sqrt[4]{1+h}$. Then $h=x^4-1$, and

$$\lim_{h\to 0}\frac{\sqrt[4]{1+h}-1}{h}=\lim_{x\to 1}\frac{x-1}{x^4-1}=\lim_{x\to 1}\frac{x-1}{(x-1)(x+1)(x^2+1)}=\lim_{x\to 1}\frac{1}{(x+1)(x^2+1)}=\frac{1}{4}$$

56. Evaluate $\lim_{h\to 0}\frac{\sqrt[3]{1+h}-1}{\sqrt[2]{1+h}-1}$. *Hint:* Set $x=\sqrt[6]{1+h}$, express h as a function of x, and rewrite as a limit as $x\to 1$.

SOLUTION Let $x=\sqrt[6]{1+h}$. The $\sqrt[3]{1+h}=x^2$, $\sqrt{1+h}=x^3$, and

$$\lim_{h\to 0}\frac{\sqrt[3]{1+h}-1}{\sqrt{1+h}-1}=\lim_{x\to 1}\frac{x^2-1}{x^3-1}=\lim_{x\to 1}\frac{(x-1)(x+1)}{(x-1)(x^2+x+1)}=\lim_{x\to 1}\frac{x+1}{x^2+x+1}=\frac{2}{3}$$

Further Insights and Challenges

In Exercises 57–60, find all values of c such that the limit exists.

57. $\lim_{x\to c} \dfrac{x^2 - 5x - 6}{x - c}$

SOLUTION $\lim_{x\to c} \dfrac{x^2 - 5x - 6}{x - c}$ will exist provided that $x - c$ is a factor of the numerator. (Otherwise there will be an infinite discontinuity at $x = c$.) Since $x^2 - 5x - 6 = (x + 1)(x - 6)$, this occurs for $c = -1$ and $c = 6$.

58. $\lim_{x\to 1} \dfrac{x^2 + 3x + c}{x - 1}$

SOLUTION $\lim_{x\to 1} \dfrac{x^2 + 3x + c}{x - 1}$ exists as long as $(x - 1)$ is a factor of $x^2 + 3x + c$. If $x^2 + 3x + c = (x - 1)(x + q)$, then $q - 1 = 3$ and $-q = c$. Hence $q = 4$ and $c = -4$.

59. $\lim_{x\to 1} \left(\dfrac{1}{x - 1} - \dfrac{c}{x^3 - 1} \right)$

SOLUTION Because $x^3 - 1 = (x - 1)(x^2 + x + 1)$,

$$\frac{1}{x - 1} - \frac{c}{x^3 - 1} = \frac{x^2 + x + 1 - c}{(x - 1)(x^2 + x + 1)}$$

Therefore, $\lim_{x\to 1} \left(\dfrac{1}{x - 1} - \dfrac{c}{x^3 - 1} \right)$ exists as long as $x - 1$ is a factor of $x^2 + x + 1 - c$. Now, if $x^2 + x + 1 - c = (x - 1)(x + q)$, then $q - 1 = 1$ and $-q = 1 - c$. Hence, $q = 2$ and $c = 3$.

60. $\lim_{x\to 0} \dfrac{1 + cx^2 - \sqrt{1 + x^2}}{x^4}$

SOLUTION Rationalizing the numerator yields

$$\frac{1 + cx^2 - \sqrt{1 + x^2}}{x^4} \left(\frac{1 + cx^2 + \sqrt{1 + x^2}}{1 + cx^2 + \sqrt{1 + x^2}} \right) = \frac{(1 + cx^2)^2 - (1 + x^2)}{x^4(\sqrt{1 + x^2} + 1 + cx^2)}$$
$$= \frac{(2c - 1)x^2 + c^2x^4}{x^4(\sqrt{1 + x^2} + 1 + cx)}$$

Therefore, $\lim_{x\to 0} \dfrac{1 + cx^2 - \sqrt{1 + x^2}}{x^4}$ exists as long as x^4 is a factor of $(2c - 1)x^2 + c^2x^4$. This will only happen if $c = \frac{1}{2}$.

61. For which sign, + or −, does the following limit exist?

$$\lim_{x\to 0} \left(\frac{1}{x} \pm \frac{1}{x(x - 1)} \right)$$

SOLUTION

- The limit $\lim_{x\to 0} \left(\dfrac{1}{x} + \dfrac{1}{x(x - 1)} \right) = \lim_{x\to 0} \dfrac{(x - 1) + 1}{x(x - 1)} = \lim_{x\to 0} \dfrac{1}{x - 1} = -1$.
- The limit $\lim_{x\to 0} \left(\dfrac{1}{x} - \dfrac{1}{x(x - 1)} \right)$ does not exist.
  - As $x \to 0+$, we have $\dfrac{1}{x} - \dfrac{1}{x(x - 1)} = \dfrac{(x - 1) - 1}{x(x - 1)} = \dfrac{x - 2}{x(x - 1)} \to \infty$.
  - As $x \to 0-$, we have $\dfrac{1}{x} - \dfrac{1}{x(x - 1)} = \dfrac{(x - 1) - 1}{x(x - 1)} = \dfrac{x - 2}{x(x - 1)} \to -\infty$.

2.6 The Squeeze Theorem and Trigonometric Limits

Preliminary Questions

1. Assume that $-x^4 \le f(x) \le x^2$. What is $\lim_{x\to 0} f(x)$? Is there enough information to evaluate $\lim_{x\to \frac{1}{2}} f(x)$? Explain.

SOLUTION Since $\lim_{x\to 0} -x^4 = \lim_{x\to 0} x^2 = 0$, the squeeze theorem guarantees that $\lim_{x\to 0} f(x) = 0$. Since $\lim_{x\to \frac{1}{2}} -x^4 = -\dfrac{1}{16} \neq \dfrac{1}{4} = \lim_{x\to \frac{1}{2}} x^2$, we do not have enough information to determine $\lim_{x\to \frac{1}{2}} f(x)$.

2. State the Squeeze Theorem carefully.

SOLUTION Assume that for $x \neq c$ (in some open interval containing c),

$$l(x) \leq f(x) \leq u(x)$$

and that $\lim_{x\to c} l(x) = \lim_{x\to c} u(x) = L$. Then $\lim_{x\to c} f(x)$ exists and

$$\lim_{x\to c} f(x) = L$$

3. If you want to evaluate $\lim_{h\to 0} \frac{\sin 5h}{3h}$, it is a good idea to rewrite the limit in terms of the variable (choose one):

(a) $\theta = 5h$ **(b)** $\theta = 3h$ **(c)** $\theta = \frac{5h}{3}$

SOLUTION To match the given limit to the pattern of

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta}$$

it is best to substitute for the argument of the sine function; thus, rewrite the limit in terms of (a): $\theta = 5h$.

Exercises

In Exercises 1–10, evaluate using the Squeeze Theorem.

1. $\lim_{x\to 0} x^2 \cos \frac{1}{x}$

SOLUTION Because $-1 \leq \cos\frac{1}{x} \leq 1$, it follows that $-x^2 \leq x^2\cos\frac{1}{x} \leq x^2$. Now, $\lim_{x\to 0}(-x^2) = \lim_{x\to 0} x^2 = 0$, so we can apply the Squeeze Theorem to conclude that

$$\lim_{x\to 0} x^2 \cos\frac{1}{x} = 0$$

2. $\lim_{x\to 0} x \sin \frac{1}{x^2}$

SOLUTION The sine function takes on values between -1 and 1; therefore, $\left|\sin\frac{1}{x^2}\right| \leq 1$ for all $x \neq 0$. Multiplying by $|x|$ yields

$$\left|x\sin\frac{1}{x^2}\right| \leq |x| \qquad \text{or} \qquad -|x| \leq x\sin\frac{1}{x^2} \leq |x|.$$

Now $\lim_{x\to 0}(-|x|) = \lim_{x\to 0}|x| = 0$, so we can apply the Squeeze Theorem to conclude that

$$\lim_{x\to 0} x\sin\frac{1}{x^2} = 0$$

3. $\lim_{x\to 1}(x-1)\sin\frac{\pi}{x-1}$

SOLUTION The sine function takes on values between -1 and 1; therefore, $\left|\sin\frac{\pi}{x-1}\right| \leq 1$ for all $x \neq 1$. Multiplying by $|x-1|$ yields

$$\left|(x-1)\sin\frac{\pi}{x-1}\right| \leq |x-1| \qquad \text{or} \qquad -|x-1| \leq (x-1)\sin\frac{\pi}{x-1} \leq |x-1|$$

Now $\lim_{x\to 1}(-|x-1|) = \lim_{x\to 1}|x-1| = 0$, so we can apply the Squeeze Theorem to conclude that

$$\lim_{x\to 1}(x-1)\sin\frac{\pi}{x-1} = 0$$

4. $\lim_{x\to 3}(x^2-9)\frac{x-3}{|x-3|}$

SOLUTION For all $x > 3$, $\frac{x-3}{|x-3|} = \frac{x-3}{x-3} = 1$, and for all $x < 3$, $\frac{x-3}{|x-3|} = \frac{x-3}{-(x-3)} = -1$; therefore, $\left|\frac{x-3}{|x-3|}\right| \leq 1$ for all $x \neq 3$. Multiplying by $|x^2-9|$ yields

$$\left|(x^2-9)\frac{x-3}{|x-3|}\right| \leq |x^2-9| \qquad \text{or} \qquad -|x^2-9| \leq (x^2-9)\frac{x-3}{|x-3|} \leq |x^2-9|$$

Now, $\lim_{x\to 3}(-|x^2-9|) = \lim_{x\to 3}|x^2-9| = 0$, so we can apply the Squeeze Theorem to conclude that

$$\lim_{x\to 3}(x^2-9)\frac{x-3}{|x-3|} = 0$$

5. $\lim_{t\to 0}(2^t-1)\cos\frac{1}{t}$

SOLUTION The cosine function takes on values between -1 and 1; therefore, $\left|\cos\frac{1}{t}\right| \le 1$ for all $t \ne 0$. Multiplying by $|2^t-1|$ yields

$$\left|(2^t-1)\cos\frac{1}{t}\right| \le |2^t-1| \qquad \text{or} \qquad -|2^t-1| \le (2^t-1)\cos\frac{1}{t} \le |2^t-1|$$

Now $\lim_{t\to 0}(-|2^t-1|) = \lim_{t\to 0}|2^t-1| = 0$, so we can apply the Squeeze Theorem to conclude that

$$\lim_{t\to 0}(2^t-1)\cos\frac{1}{t} = 0$$

6. $\lim_{x\to 0^+}\sqrt{x}\,3^{\cos(\pi/x)}$

SOLUTION Because $-1 \le \cos\frac{\pi}{x} \le 1$ and 3^x is an increasing function, it follows that

$$\frac{1}{3} \le 3^{\cos(\pi/x)} \le 3 \quad \text{and} \quad \frac{1}{3}\sqrt{x} \le \sqrt{x}\,3^{\cos(\pi/x)} \le 3\sqrt{x}$$

Now

$$\lim_{x\to 0^+}\frac{1}{3}\sqrt{x} = \lim_{x\to 0^+}3\sqrt{x} = 0$$

so we can apply the Squeeze Theorem to conclude that

$$\lim_{x\to 0^+}\sqrt{x}\,3^{\cos(\pi/x)} = 0$$

7. $\lim_{t\to 2}(t^2-4)\cos\frac{1}{t-2}$

SOLUTION The cosine function takes on values between -1 and 1; therefore, $\left|\cos\frac{1}{t-2}\right| \le 1$ for all $t \ne 2$. Multiplying by $|t^2-4|$ yields

$$\left|(t^2-4)\cos\frac{1}{t-2}\right| \le |t^2-4| \qquad \text{or} \qquad -|t^2-4| \le (t^2-4)\cos\frac{1}{t-2} \le |t^2-4|$$

Now $\lim_{t\to 2}(-|t^2-4|) = \lim_{t\to 2}|t^2-4| = 0$, so we can apply the Squeeze Theorem to conclude that

$$\lim_{t\to 2}(t^2-4)\cos\frac{1}{t-2} = 0$$

8. $\lim_{x\to 0}\tan x\cos\left(\sin\frac{1}{x}\right)$

SOLUTION The cosine function takes on values between -1 and 1; therefore, $\left|\cos\left(\sin\frac{1}{x}\right)\right| \le 1$ for all $x \ne 0$. Multiplying by $|\tan x|$ yields

$$\left|\tan x\cos\left(\sin\frac{1}{x}\right)\right| \le |\tan x| \qquad \text{or} \qquad -|\tan x| \le \tan x\cos\left(\sin\frac{1}{x}\right) \le |\tan x|$$

Now $\lim_{x\to 0}(-|\tan x|) = \lim_{x\to 0}|\tan x| = 0$, so we can apply the Squeeze Theorem to conclude that

$$\lim_{x\to 0}\tan x\cos\left(\sin\frac{1}{x}\right) = 0$$

9. $\lim_{\theta\to\frac{\pi}{2}}\cos\theta\cos(\tan\theta)$

SOLUTION The cosine function takes on values between −1 and 1; therefore, $|\cos(\tan\theta)| \le 1$ for all θ near $\frac{\pi}{2}$. Multiplying by $|\cos\theta|$ yields

$$|\cos\theta\cos(\tan\theta)| \le |\cos\theta| \qquad \text{or} \qquad -|\cos\theta| \le \cos\theta\cos(\tan\theta) \le |\cos\theta|$$

Now $\lim_{\theta\to\frac{\pi}{2}}(-|\cos\theta|) = \lim_{\theta\to\frac{\pi}{2}}|\cos\theta| = 0$, so we can apply the Squeeze Theorem to conclude that

$$\lim_{\theta\to\frac{\pi}{2}}\cos\theta\cos(\tan\theta) = 0$$

10. $\lim_{t\to0^-}(3^t - 1)\sin^2\left(\frac{1}{t}\right)$

SOLUTION For all t near 0 but less than 0, $0 \le \sin^2\left(\frac{1}{t}\right) \le 1$ and $3^t - 1 < 0$. Therefore, for t near 0 but less than 0,

$$3^t - 1 \le (3^t - 1)\sin^2\left(\frac{1}{t}\right) \le 0$$

Now, $\lim_{t\to0^-}(3^t - 1) = 0$ and $\lim_{t\to0^-} 0 = 0$, so we can apply the Squeeze Theorem to conclude that

$$\lim_{t\to0^-}(3^t - 1)\sin^2\left(\frac{1}{t}\right) = 0.$$

11. State precisely the hypothesis and conclusions of the Squeeze Theorem for the situation in Figure 6.

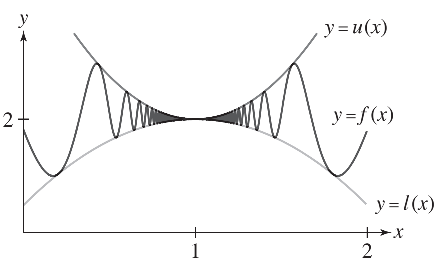


FIGURE 6

SOLUTION Because there is an open interval containing $x = 1$ on which $l(x) \le f(x) \le u(x)$ and $\lim_{x\to1} l(x) = \lim_{x\to1} u(x) = 2$, it follows that $\lim_{x\to1} f(x)$ exists and

$$\lim_{x\to1} f(x) = 2$$

12. In Figure 7, is f squeezed by u and l at $x = 3$? At $x = 2$?

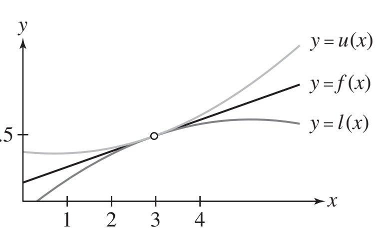


FIGURE 7

SOLUTION Because there is an open interval containing $x = 3$ on which $l(x) \le f(x) \le u(x)$ and $\lim_{x\to3} l(x) = \lim_{x\to3} u(x)$, $f(x)$ is *squeezed* by $u(x)$ and $l(x)$ at $x = 3$. Because there is an open interval containing $x = 2$ on which $l(x) \le f(x) \le u(x)$ but $\lim_{x\to2} l(x) \ne \lim_{x\to2} u(x)$, $f(x)$ is *trapped* by $u(x)$ and $l(x)$ at $x = 2$ but not *squeezed*.

13. What does the Squeeze Theorem say about $\lim_{x\to7} f(x)$ if the limits $\lim_{x\to7} l(x) = \lim_{x\to7} u(x) = 6$ and f, u, and l are related as in Figure 8? The inequality $f(x) \le u(x)$ is not satisfied for all x. Does this affect the validity of your conclusion?

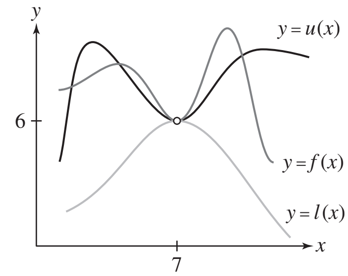


FIGURE 8

SOLUTION The Squeeze Theorem does not require that the inequalities $l(x) \le f(x) \le u(x)$ hold for all x, only that the inequalities hold on some open interval containing $x = c$. In Figure 8, it is clear that $l(x) \le f(x) \le u(x)$ on some open interval containing $x = 7$. Because $\lim_{x\to 7} u(x) = \lim_{x\to 7} l(x) = 6$, the Squeeze Theorem guarantees that $\lim_{x\to 7} f(x) = 6$.

14. Determine $\lim_{x\to 0} f(x)$ assuming that $\cos x \le f(x) \le 1$.

SOLUTION By the Squeeze Theorem, $\lim_{x\to 0} \cos x \le \lim_{x\to 0} f(x) \le \lim_{x\to 0} 1$. Hence, $1 \le \lim_{x\to 0} f(x) \le 1$, so $\lim_{x\to 0} f(x) = 1$.

15. State whether the inequality provides sufficient information to determine $\lim_{x\to 1} f(x)$, and if so, find the limit.

(a) $4x - 5 \le f(x) \le x^2$
(b) $2x - 1 \le f(x) \le x^2$
(c) $4x - x^2 \le f(x) \le x^2 + 2$

SOLUTION

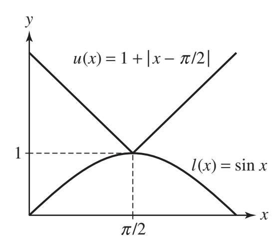
(a) Because $\lim_{x\to 1}(4x - 5) = -1 \ne 1 = \lim_{x\to 1} x^2$, the given inequality does *not* provide sufficient information to determine $\lim_{x\to 1} f(x)$.

(b) Because $\lim_{x\to 1}(2x - 1) = 1 = \lim_{x\to 1} x^2$, it follows from the Squeeze Theorem that $\lim_{x\to 1} f(x) = 1$.

(c) Because $\lim_{x\to 1}(4x - x^2) = 3 = \lim_{x\to 1}(x^2 + 2)$, it follows from the Squeeze Theorem that $\lim_{x\to 1} f(x) = 3$.

16. GU Plot the graphs of $u(x) = 1 + \left|x - \frac{\pi}{2}\right|$ and $l(x) = \sin x$ on the same set of axes. What can you say about $\lim_{x\to \frac{\pi}{2}} f(x)$ if f is squeezed by l and u at $x = \frac{\pi}{2}$?

SOLUTION



$\lim_{x\to\pi/2} u(x) = 1$ and $\lim_{x\to\pi/2} l(x) = 1$, so any function $f(x)$ satisfying $l(x) \le f(x) \le u(x)$ for all x near $\pi/2$ will satisfy $\lim_{x\to\pi/2} f(x) = 1$.

In Exercises 17–26, evaluate using Theorem 2 as necessary.

17. $\lim_{x\to 0} \frac{\tan x}{x}$

SOLUTION $\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\sin x}{x} \frac{1}{\cos x} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{1}{\cos x} = 1 \cdot 1 = 1$

18. $\lim_{x\to 0} \frac{\sin x \sec x}{x}$

SOLUTION $\lim_{x\to 0} \frac{\sin x \sec x}{x} = \lim_{x\to 0} \frac{\sin x}{x} \sec x = \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \sec x = 1 \cdot 1 = 1$

19. $\lim_{t\to 0} \frac{\sqrt{t^3 + 9}\sin t}{t}$

SOLUTION $\lim_{t\to 0} \frac{\sqrt{t^3 + 9}\sin t}{t} = \lim_{t\to 0} \sqrt{t^3 + 9}\frac{\sin t}{t} = \lim_{t\to 0} \sqrt{t^3 + 9} \cdot \lim_{t\to 0} \frac{\sin t}{t} = \sqrt{9} \cdot 1 = 3$

20. $\lim_{t\to 0} \frac{\sin^2 t}{t}$

SOLUTION $\lim_{t\to 0} \frac{\sin^2 t}{t} = \lim_{t\to 0} \frac{\sin t}{t} \sin t = \lim_{t\to 0} \frac{\sin t}{t} \cdot \lim_{t\to 0} \sin t = 1 \cdot 0 = 0$

21. $\lim_{x\to 0} \frac{x^2}{\sin^2 x}$

SOLUTION $\lim_{x\to 0} \frac{x^2}{\sin^2 x} = \lim_{x\to 0} \frac{1}{\frac{\sin x}{x}\,\frac{\sin x}{x}} = \lim_{x\to 0} \frac{1}{\frac{\sin x}{x}} \cdot \lim_{x\to 0} \frac{1}{\frac{\sin x}{x}} = \frac{1}{1} \cdot \frac{1}{1} = 1$

22. $\lim_{t\to \frac{\pi}{2}} \frac{1-\cos t}{t}$

SOLUTION The function $\frac{1-\cos t}{t}$ is continuous at $\frac{\pi}{2}$; evaluate using substitution:

$$\lim_{t\to \frac{\pi}{2}} \frac{1-\cos t}{t} = \frac{1-0}{\frac{\pi}{2}} = \frac{2}{\pi}$$

23. $\lim_{\theta\to 0} \frac{\sec\theta - 1}{\theta}$

SOLUTION $\lim_{\theta\to 0} \frac{\sec\theta - 1}{\theta} = \lim_{\theta\to 0} \frac{\sec\theta - 1}{\theta}\frac{\cos\theta}{\cos\theta} = \lim_{\theta\to 0} \frac{1-\cos\theta}{\theta}\frac{1}{\cos\theta} = \lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} \cdot \lim_{\theta\to 0} \frac{1}{\cos\theta} = 0 \cdot \frac{1}{1} = 0$

24. $\lim_{\theta\to 0} \frac{1-\cos\theta}{\sin\theta}$

SOLUTION $\lim_{\theta\to 0} \frac{1-\cos\theta}{\sin\theta} = \lim_{\theta\to 0} \frac{1-\cos\theta}{\theta}\frac{\theta}{\sin\theta} = \lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} \cdot \lim_{\theta\to 0} \frac{1}{\frac{\sin\theta}{\theta}} = 0 \cdot \frac{1}{1} = 0$

25. $\lim_{t\to \frac{\pi}{4}} \frac{\sin t}{t}$

SOLUTION $\frac{\sin t}{t}$ is continuous at $t = \frac{\pi}{4}$. Hence, by substitution

$$\lim_{t\to \frac{\pi}{4}} \frac{\sin t}{t} = \frac{\frac{\sqrt{2}}{2}}{\frac{\pi}{4}} = \frac{2\sqrt{2}}{\pi}$$

26. $\lim_{t\to 0} \frac{\cos t - \cos^2 t}{t}$

SOLUTION By factoring and applying the Product Law:

$$\lim_{t\to 0} \frac{\cos t - \cos^2 t}{t} = \lim_{t\to 0} \cos t \cdot \lim_{t\to 0} \frac{1-\cos t}{t} = 1(0) = 0$$

27. Evaluate $\lim_{x\to 0} \frac{\sin 11x}{x}$ using a substitution $\theta = 11x$.

SOLUTION Let $\theta = 11x$. Then $\theta \to 0$ as $x \to 0$, and $x = \theta/11$. Hence,

$$\lim_{x\to 0} \frac{\sin 11x}{x} = \lim_{\theta\to 0} \frac{\sin\theta}{\theta/11} = 11 \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 11(1) = 11$$

28. Evaluate $\lim_{t\to 0} \frac{\sin 7t}{\sin 11t}$. *Hint:* Multiply the numerator and denominator by $(7)(11)t$.

SOLUTION Following the hint,

$$\lim_{t\to 0} \frac{\sin 7t}{\sin 11t} = \lim_{t\to 0} \frac{\sin 7t}{\sin 11t} \cdot \frac{7(11)t}{7(11)t} = \lim_{t\to 0} \frac{\sin 7t}{7t} \cdot \frac{11t}{\sin 11t} \cdot \frac{7}{11}$$
$$= \frac{7}{11} \lim_{t\to 0} \frac{\sin 7t}{7t} \lim_{t\to 0} \frac{11t}{\sin 11t} = \frac{7}{11}(1)(1) = \frac{7}{11}$$

In Exercises 29–48, evaluate the limit.

29. $\lim_{h\to 0} \frac{\sin 9h}{h}$

SOLUTION $\lim_{h\to 0} \frac{\sin 9h}{h} = \lim_{h\to 0} 9\frac{\sin 9h}{9h} = 9.$

30. $\lim_{h\to 0} \frac{\sin 4h}{4h}$

SOLUTION Let $\theta = 4h$. Then $\theta \to 0$ as $h \to 0$, and

$$\lim_{h\to 0} \frac{\sin 4h}{4h} = \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

31. $\displaystyle\lim_{h\to 0} \frac{\sin h}{5h}$

SOLUTION $\displaystyle\lim_{h\to 0} \frac{\sin h}{5h} = \lim_{h\to 0} \frac{1}{5}\frac{\sin h}{h} = \frac{1}{5}$

32. $\displaystyle\lim_{x\to \frac{\pi}{6}} \frac{x}{\sin 3x}$

SOLUTION $\displaystyle\lim_{x\to \frac{\pi}{6}} \frac{x}{\sin 3x} = \frac{\pi/6}{\sin(\pi/2)} = \frac{\pi}{6}$

33. $\displaystyle\lim_{\theta\to 0} \frac{\sin 7\theta}{\sin 3\theta}$

SOLUTION We have

$$\frac{\sin 7\theta}{\sin 3\theta} = \frac{7}{3}\left(\frac{\sin 7\theta}{7\theta}\right)\left(\frac{3\theta}{\sin 3\theta}\right)$$

Therefore,

$$\lim_{\theta\to 0} \frac{\sin 7\theta}{3\theta} = \frac{7}{3}\left(\lim_{\theta\to 0}\frac{\sin 7\theta}{7\theta}\right)\left(\lim_{\theta\to 0}\frac{3\theta}{\sin 3\theta}\right) = \frac{7}{3}(1)(1) = \frac{7}{3}$$

34. $\displaystyle\lim_{x\to 0} \frac{\tan 4x}{9x}$

SOLUTION $\displaystyle\lim_{x\to 0} \frac{\tan 4x}{9x} = \lim_{x\to 0} \frac{1}{9}\cdot\frac{\sin 4x}{4x}\cdot\frac{4}{\cos 4x} = \frac{4}{9}$

35. $\displaystyle\lim_{x\to 0} x \csc 25x$

SOLUTION $\displaystyle\lim_{x\to 0} x\csc 25x = \lim_{x\to 0} \frac{x}{\sin 25x} = \frac{1}{25}\lim_{x\to 0}\frac{25x}{\sin 25x} = \frac{1}{25}$

36. $\displaystyle\lim_{t\to 0} \frac{\tan 4t}{t\sec t}$

SOLUTION $\displaystyle\lim_{t\to 0} \frac{\tan 4t}{t\sec t} = \lim_{t\to 0} \frac{4\sin 4t}{4t\cos(4t)\sec(t)} = \lim_{t\to 0}\frac{4\cos t}{\cos 4t}\cdot\frac{\sin 4t}{4t} = 4$

37. $\displaystyle\lim_{h\to 0} \frac{\sin 2h\ \sin 3h}{h^2}$

SOLUTION

$$\begin{aligned}\lim_{h\to 0} \frac{\sin 2h\ \sin 3h}{h^2} &= \lim_{h\to 0}\frac{\sin 2h\ \sin 3h}{h\cdot h} = \lim_{h\to 0}\frac{\sin 2h}{h}\ \frac{\sin 3h}{h}\\ &= \lim_{h\to 0} 2\frac{\sin 2h}{2h}\ 3\frac{\sin 3h}{3h} = \lim_{h\to 0} 2\frac{\sin 2h}{2h}\ \lim_{h\to 0} 3\frac{\sin 3h}{3h} = 2\cdot 3 = 6\end{aligned}$$

38. $\displaystyle\lim_{z\to 0} \frac{\sin(z/3)}{\sin z}$

SOLUTION $\displaystyle\lim_{z\to 0} \frac{\sin(z/3)}{\sin z}\cdot\frac{z/3}{z/3} = \lim_{z\to 0}\frac{1}{3}\cdot\frac{z}{\sin z}\cdot\frac{\sin(z/3)}{z/3} = \frac{1}{3}$

39. $\displaystyle\lim_{\theta\to 0} \frac{\sin(-3\theta)}{\sin 4\theta}$

SOLUTION $\displaystyle\lim_{\theta\to 0} \frac{\sin(-3\theta)}{\sin(4\theta)} = \lim_{\theta\to 0}\frac{-\sin(3\theta)}{3\theta}\cdot\frac{3}{4}\cdot\frac{4\theta}{\sin(4\theta)} = -\frac{3}{4}$

40. $\displaystyle\lim_{x\to 0} \frac{\tan 4x}{\tan 9x}$

SOLUTION $\displaystyle\lim_{x\to 0} \frac{\tan 4x}{\tan 9x} = \lim_{x\to 0}\frac{\cos 9x}{\cos 4x}\cdot\frac{\sin 4x}{4x}\cdot\frac{4}{9}\cdot\frac{9x}{\sin 9x} = \frac{4}{9}$

41. $\displaystyle\lim_{t\to 0} \frac{\csc 8t}{\csc 4t}$

SOLUTION $\lim_{t\to 0} \frac{\csc 8t}{\csc 4t} = \lim_{t\to 0} \frac{\sin 4t}{\sin 8t} \cdot \frac{8t}{4t} \cdot \frac{1}{2} = \frac{1}{2}$

42. $\lim_{x\to 0} \frac{\sin 5x \sin 2x}{\sin 3x \sin 5x}$

SOLUTION $\lim_{x\to 0} \frac{\sin 5x \sin 2x}{\sin 3x \sin 5x} = \lim_{x\to 0} \frac{\sin 2x}{2x} \cdot \frac{2}{3} \cdot \frac{3x}{\sin 3x} = \frac{2}{3}$

43. $\lim_{x\to 0} \frac{\sin 3x \sin 2x}{x \sin 5x}$

SOLUTION $\lim_{x\to 0} \frac{\sin 3x \sin 2x}{x \sin 5x} = \lim_{x\to 0} \left(3 \frac{\sin 3x}{3x} \cdot \frac{2}{5} \frac{(\sin 2x)/(2x)}{(\sin 5x)/(5x)}\right) = \frac{6}{5}$

44. $\lim_{h\to 0} \frac{1-\cos 2h}{h}$

SOLUTION $\lim_{h\to 0} \frac{1-\cos 2h}{h} = \lim_{h\to 0} 2\frac{1-\cos 2h}{2h} = 2\lim_{h\to 0} \frac{1-\cos 2h}{2h} = 2\cdot 0 = 0$

45. $\lim_{h\to 0} \frac{\sin(2h)(1-\cos h)}{h^2}$

SOLUTION $\lim_{h\to 0} \frac{\sin(2h)(1-\cos h)}{h^2} = \lim_{h\to 0} 2\frac{\sin(2h)}{2h} \lim_{h\to 0} \frac{1-\cos h}{h} = 2\cdot 0 = 0$

46. $\lim_{t\to 0} \frac{1-\cos 2t}{\sin^2 3t}$

SOLUTION Using the identity $\cos 2t = 1 - 2\sin^2 t$,

$$\lim_{t\to 0} \frac{1-\cos 2t}{\sin^2 3t} = \lim_{t\to 0} \frac{2\sin^2 t}{\sin^2 3t} = \lim_{t\to 0} \frac{2}{9} \cdot \frac{\sin t}{t} \cdot \frac{\sin t}{t} \cdot \frac{3t}{\sin 3t} \cdot \frac{3t}{\sin 3t} = \frac{2}{9}$$

47. $\lim_{\theta\to 0} \frac{\cos 2\theta - \cos\theta}{\theta}$

SOLUTION

$$\lim_{\theta\to 0} \frac{\cos 2\theta - \cos\theta}{\theta} = \lim_{\theta\to 0} \frac{(\cos 2\theta - 1) + (1-\cos\theta)}{\theta} = \lim_{\theta\to 0} \frac{\cos 2\theta - 1}{\theta} + \lim_{\theta\to 0} \frac{1-\cos\theta}{\theta}$$
$$= -2\lim_{\theta\to 0} \frac{1-\cos 2\theta}{2\theta} + \lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} = -2\cdot 0 + 0 = 0$$

48. $\lim_{h\to \frac{\pi}{2}} \frac{1-\cos 3h}{h}$

SOLUTION The function is continuous at $\frac{\pi}{2}$, so we may use substitution:

$$\lim_{h\to \frac{\pi}{2}} \frac{1-\cos 3h}{h} = \frac{1-\cos\frac{3\pi}{2}}{\frac{\pi}{2}} = \frac{1-0}{\frac{\pi}{2}} = \frac{2}{\pi}$$

49. Use the identity $\sin 2\theta = 2\sin\theta\cos\theta$ to evaluate $\lim_{\theta\to 0} \frac{\sin 2\theta - 2\sin\theta}{\theta^2}$.

SOLUTION Using the identity $\sin 2\theta = 2\sin\theta\cos\theta$,

$$\sin 2\theta - 2\sin\theta = 2\sin\theta\cos\theta - 2\sin\theta = 2\sin\theta(\cos\theta - 1) = -2\sin\theta(1-\cos\theta)$$

Then

$$\lim_{\theta\to 0} \frac{\sin 2\theta - 2\sin\theta}{\theta^2} = -2\lim_{\theta\to 0} \frac{\sin\theta(\cos\theta - 1)}{\theta^2} = -2\lim_{\theta\to 0} \frac{\sin\theta}{\theta} \cdot \frac{1-\cos\theta}{\theta} = -2(1)(0) = 0$$

50. Use the identity $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ to evaluate $\lim_{\theta\to 0} \frac{\sin 3\theta - 3\sin\theta}{\theta^3}$.

SOLUTION Using the identity $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$,

$$\sin 3\theta - 3\sin\theta = 3\sin\theta - 4\sin^3\theta - 3\sin\theta = -4\sin^3\theta$$

Then

$$\lim_{\theta\to 0} \frac{\sin 3\theta - 3\sin\theta}{\theta^3} = -4\lim_{\theta\to 0} \frac{\sin^3\theta}{\theta^3} = -4\lim_{\theta\to 0} \frac{\sin\theta}{\theta} \cdot \frac{\sin\theta}{\theta} \cdot \frac{\sin\theta}{\theta} = -4$$

51. Explain why $\lim_{\theta\to 0}(\csc\theta - \cot\theta)$ involves an indeterminate form, and then prove that the limit equals 0.

SOLUTION As θ approaches 0 from the right, $\csc\theta \to \infty$ and $\cot\theta \to \infty$, and as θ approaches 0 from the left, $\csc\theta \to -\infty$ and $\cot\theta \to -\infty$. Thus, $\csc\theta - \cot\theta$ has the indeterminate form $\infty - \infty$ as $\theta \to 0$.

Now,

$$\csc\theta - \cot\theta = \frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta} = \frac{1-\cos\theta}{\sin\theta}$$

so

$$\lim_{\theta\to 0}(\csc\theta - \cot\theta) = \lim_{\theta\to 0}\frac{1-\cos\theta}{\sin\theta} = \lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}\,\frac{\theta}{\sin\theta} = 0\cdot 1 = 0$$

52. Explain why $\lim_{\theta\to\frac{\pi}{2}}(2\tan\theta - \sec\theta)$ involves an indeterminate form, and then evaluate the limit.

SOLUTION As θ approaches $\pi/2$ from the left, $2\tan\theta \to \infty$ and $\sec\theta \to \infty$, and as θ approaches $\pi/2$ from the right, $2\tan\theta \to -\infty$ and $\sec\theta \to -\infty$. Thus, $2\tan\theta - \sec\theta$ has the indeterminate form $\infty - \infty$ as $\theta \to \pi/2$.

Now,

$$\lim_{\theta\to\pi/2}(2\tan\theta - \sec\theta) = \lim_{\theta\to\pi/2}\left(2\frac{\sin\theta}{\cos\theta} - \frac{1}{\cos\theta}\right) = \lim_{\theta\to\pi/2}\frac{2\sin\theta - 1}{\cos\theta}$$

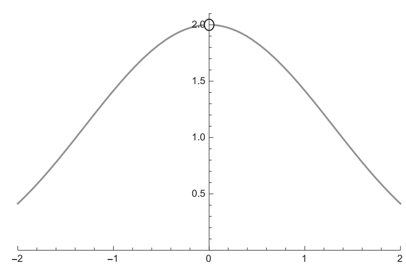
Because $2\sin\theta - 1 \to 1 \neq 0$ but $\cos\theta \to 0$ as $\theta \to \pi/2$, it follows that the requested limit does not exist.

53. GU Investigate $\lim_{h\to 0}\frac{1-\cos 2h}{h^2}$ numerically or graphically. Then evaluate the limit using the double angle formula $\cos 2h = 1 - 2\sin^2 h$.

SOLUTION

h	−0.1	−0.01	0.01	0.1
$\frac{1-\cos 2h}{h^2}$	1.993342	1.999933	1.999933	1.993342

•



Both the numerical estimates and the graph suggest that the value of the limit is 2.

- Using the double angle formula $\cos 2h = 1 - 2\sin^2 h$,

$$1 - \cos 2h = 1 - (1 - 2\sin^2 h) = 2\sin^2 h$$

Then

$$\lim_{h\to 0}\frac{1-\cos 2h}{h^2} = \lim_{h\to 0}\frac{2\sin^2 h}{h^2} = 2\lim_{h\to 0}\frac{\sin h}{h}\cdot\frac{\sin h}{h} = 2(1)(1) = 2$$

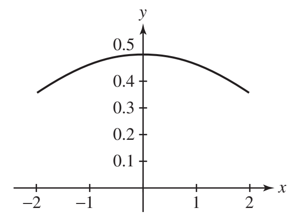
54. GU Investigate $\lim_{h\to 0}\frac{1-\cos h}{h^2}$ numerically or graphically. Then prove that the limit is equal to $\frac{1}{2}$. *Hint:* See the proof of Theorem 2.

SOLUTION

•

h	−.1	−.01	.01	.1
$\frac{1-\cos h}{h^2}$	.499583	.499996	.499996	.499583

•



Both the numerical estimates and the graph suggest that the value of the limit is $\frac{1}{2}$.

• $\displaystyle \lim_{h\to 0} \frac{1-\cos h}{h^2} = \lim_{h\to 0} \frac{1-\cos^2 h}{h^2(1+\cos h)} = \lim_{h\to 0} \left(\frac{\sin h}{h}\right)^2 \frac{1}{1+\cos h} = \frac{1}{2}$

In Exercises 55–57, evaluate using the result of Exercise 54.

55. $\displaystyle \lim_{h\to 0} \frac{\cos 3h - 1}{h^2}$

SOLUTION We make the substitution $\theta = 3h$. Then $h = \theta/3$, and

$$\lim_{h\to 0} \frac{\cos 3h - 1}{h^2} = \lim_{\theta\to 0} \frac{\cos\theta - 1}{(\theta/3)^2} = -9 \lim_{\theta\to 0} \frac{1-\cos\theta}{\theta^2} = -\frac{9}{2}$$

56. $\displaystyle \lim_{h\to 0} \frac{\cos 3h - 1}{\cos 2h - 1}$

SOLUTION $\displaystyle \lim_{h\to 0} \frac{\cos 3h - 1}{\cos 2h - 1} = \lim_{h\to 0} \frac{1-\cos 3h}{1-\cos 2h} = \lim_{h\to 0} \frac{1-\cos 3h}{(3h)^2} \cdot \frac{3^2}{2^2} \cdot \frac{(2h)^2}{1-\cos 2h} = \frac{1}{2} \cdot \frac{9}{4} \cdot \frac{1}{1/2} = \frac{9}{4}$

57. $\displaystyle \lim_{t\to 0} \frac{\sqrt{1-\cos t}}{t}$

SOLUTION $\displaystyle \lim_{t\to 0} \frac{\sqrt{1-\cos t}}{t} = \sqrt{\lim_{t\to 0} \frac{1-\cos t}{t^2}} = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$

58. Use the Squeeze Theorem to prove that if $\lim_{x\to c} |f(x)| = 0$, then $\lim_{x\to c} f(x) = 0$.

SOLUTION Suppose $\lim_{x\to c} |f(x)| = 0$. Then

$$\lim_{x\to c} -|f(x)| = -\lim_{x\to c} |f(x)| = 0$$

Now, for all x, the inequalities

$$-|f(x)| \le f(x) \le |f(x)|$$

hold. Because $\lim_{x\to c} |f(x)| = 0$ and $\lim_{x\to c} -|f(x)| = 0$, it follows from the Squeeze Theorem that $\lim_{x\to c} f(x) = 0$.

Further Insights and Challenges

59. Use the result of Exercise 54 to prove that for $m \neq 0$,

$$\lim_{x\to 0} \frac{\cos mx - 1}{x^2} = -\frac{m^2}{2}$$

SOLUTION Substitute $u = mx$ into $\dfrac{\cos mx - 1}{x^2}$. We obtain $x = \frac{u}{m}$. As $x \to 0$, $u \to 0$; therefore,

$$\lim_{x\to 0} \frac{\cos mx - 1}{x^2} = \lim_{u\to 0} \frac{\cos u - 1}{(u/m)^2} = \lim_{u\to 0} m^2 \frac{\cos u - 1}{u^2} = m^2\left(-\frac{1}{2}\right) = -\frac{m^2}{2}$$

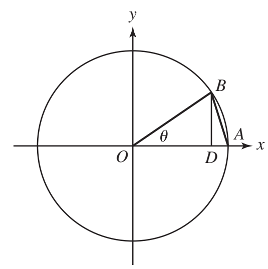
60. Using a diagram of the unit circle and the Pythagorean Theorem, show that

$$\sin^2\theta \le (1-\cos\theta)^2 + \sin^2\theta \le \theta^2$$

Conclude that $\sin^2\theta \le 2(1-\cos\theta) \le \theta^2$ and use this to give an alternative proof that the limit in Exercise 51 equals 0. Then give an alternative proof of the result in Exercise 54.

SOLUTION

- Consider the unit circle shown below. The triangle BDA is a right triangle. It has base $1 - \cos\theta$, altitude $\sin\theta$, and hypotenuse h. Observe that the hypotenuse h is less than the arc length AB = radius · angle $= 1 \cdot \theta = \theta$. Apply the Pythagorean Theorem to obtain $(1 - \cos\theta)^2 + \sin^2\theta = h^2 \leq \theta^2$. The inequality $\sin^2\theta \leq (1 - \cos\theta)^2 + \sin^2\theta$ follows from the fact that $(1 - \cos\theta)^2 \geq 0$.



- Note that

$$(1 - \cos\theta)^2 + \sin^2\theta = 1 - 2\cos\theta + \cos^2\theta + \sin^2\theta = 2 - 2\cos\theta = 2(1 - \cos\theta)$$

Therefore,

$$\sin^2\theta \leq 2(1 - \cos\theta) \leq \theta^2$$

- Divide the previous inequality by $2\sin\theta$ to obtain

$$\frac{\sin\theta}{2} \leq \frac{1 - \cos\theta}{\sin\theta} = \csc\theta - \cot\theta \leq \frac{\theta^2}{2\sin\theta}$$

Because $\lim_{\theta\to 0} \frac{\sin\theta}{2} = 0$ and

$$\lim_{\theta\to 0} \frac{\theta^2}{2\sin\theta} = \frac{1}{2}\lim_{\theta\to 0} \frac{\theta}{\sin\theta} \cdot \theta = \frac{1}{2}(1)(0) = 0$$

it follows by the Squeeze Theorem that

$$\lim_{\theta\to 0}(\csc\theta - \cot\theta) = 0$$

- Divide the inequality

$$\sin^2\theta \leq 2(1 - \cos\theta) \leq \theta^2$$

by $2\theta^2$ to obtain

$$\frac{\sin^2\theta}{2\theta^2} \leq \frac{1 - \cos\theta}{\theta^2} \leq \frac{1}{2}$$

Because

$$\lim_{\theta\to 0} \frac{\sin^2\theta}{2\theta^2} = \frac{1}{2}\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)^2 = \frac{1}{2}(1^2) = \frac{1}{2}$$

and $\lim_{\theta\to 0} \frac{1}{2} = \frac{1}{2}$, it follows by the Squeeze Theorem that

$$\lim_{\theta\to 0} \frac{1 - \cos\theta}{\theta^2} = \frac{1}{2}$$

61. (a) Investigate $\lim_{x\to c} \frac{\sin x - \sin c}{x - c}$ numerically for the five values $c = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$.

(b) Can you guess the answer for general c?

(c) Check numerically that your answer to (b) works for two other values of c.

SOLUTION

(a) Here $c = 0$ and $\cos c = 1$.

x	$c - 0.01$	$c - 0.001$	$c + 0.001$	$c + 0.01$
$\dfrac{\sin x - \sin c}{x - c}$	0.999983	0.99999983	0.99999983	0.999983

Here $c = \frac{\pi}{6}$ and $\cos c = \frac{\sqrt{3}}{2} \approx .866025$.

x	$c - 0.01$	$c - 0.001$	$c + 0.001$	$c + 0.01$
$\dfrac{\sin x - \sin c}{x - c}$	0.868511	0.866275	0.865775	0.863511

Here $c = \frac{\pi}{3}$ and $\cos c = \frac{1}{2}$.

x	$c - 0.01$	$c - 0.001$	$c + 0.001$	$c + 0.01$
$\dfrac{\sin x - \sin c}{x - c}$	0.504322	0.500433	0.499567	0.495662

Here $c = \frac{\pi}{4}$ and $\cos c = \frac{\sqrt{2}}{2} \approx 0.707107$.

x	$c - 0.01$	$c - 0.001$	$c + 0.001$	$c + 0.01$
$\dfrac{\sin x - \sin c}{x - c}$	0.710631	0.707460	0.706753	0.703559

Here $c = \frac{\pi}{2}$ and $\cos c = 0$.

x	$c - 0.01$	$c - 0.001$	$c + 0.001$	$c + 0.01$
$\dfrac{\sin x - \sin c}{x - c}$	0.005000	0.000500	−0.000500	−0.005000

(b) $\displaystyle\lim_{x \to c} \frac{\sin x - \sin c}{x - c} = \cos c.$

(c) Here $c = 2$ and $\cos c = \cos 2 \approx -.416147$.

x	$c - 0.01$	$c - 0.001$	$c + 0.001$	$c + 0.01$
$\dfrac{\sin x - \sin c}{x - c}$	−0.411593	−0.415692	−0.416601	−0.420686

Here $c = -\frac{\pi}{6}$ and $\cos c = \frac{\sqrt{3}}{2} \approx .866025$.

x	$c - 0.01$	$c - 0.001$	$c + 0.001$	$c + 0.01$
$\dfrac{\sin x - \sin c}{x - c}$	0.863511	0.865775	0.866275	0.868511

2.7 Limits at Infinity

Preliminary Questions

1. Assume that

$$\lim_{x \to \infty} f(x) = L \quad \text{and} \quad \lim_{x \to L} g(x) = \infty$$

Which of the following statements are correct?

(a) $x = L$ is a vertical asymptote of g.

(b) $y = L$ is a horizontal asymptote of g.

(c) $x = L$ is a vertical asymptote of f.
(d) $y = L$ is a horizontal asymptote of f.

SOLUTION

(a) Because $\lim_{x\to L} g(x) = \infty$, $x = L$ is a vertical asymptote of $g(x)$. This statement is correct.
(b) This statement is not correct.
(c) This statement is not correct.
(d) Because $\lim_{x\to\infty} f(x) = L$, $y = L$ is a horizontal asymptote of $f(x)$. This statement is correct.

2. What are the following limits?

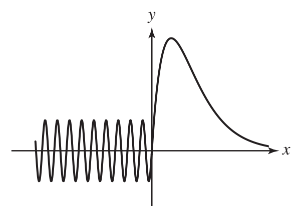
(a) $\lim_{x\to\infty} x^3$ **(b)** $\lim_{x\to-\infty} x^3$ **(c)** $\lim_{x\to-\infty} x^4$

SOLUTION

(a) $\lim_{x\to\infty} x^3 = \infty$
(b) $\lim_{x\to-\infty} x^3 = -\infty$
(c) $\lim_{x\to-\infty} x^4 = \infty$

3. Sketch the graph of a function that approaches a limit as $x \to \infty$ but does not approach a limit (either finite or infinite) as $x \to -\infty$.

SOLUTION



4. What is the sign of a if $f(x) = ax^3 + x + 1$ satisfies $\lim_{x\to-\infty} f(x) = \infty$?

SOLUTION Because $\lim_{x\to-\infty} x^3 = -\infty$, a must be negative to have $\lim_{x\to-\infty} f(x) = \infty$.

5. What is the sign of the coefficient multiplying x^7 if f is a polynomial of degree 7 such that $\lim_{x\to-\infty} f(x) = \infty$?

SOLUTION The behavior of $f(x)$ as $x \to -\infty$ is controlled by the leading term; that is, $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} a_7x^7$. Because $x^7 \to -\infty$ as $x \to -\infty$, a_7 must be negative to have $\lim_{x\to-\infty} f(x) = \infty$.

6. Explain why $\lim_{x\to\infty} \sin\frac{1}{x}$ exists but $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist. What is $\lim_{x\to\infty} \sin\frac{1}{x}$?

SOLUTION As $x \to \infty$, $\frac{1}{x} \to 0$, so

$$\lim_{x\to\infty} \sin\frac{1}{x} = \sin 0 = 0$$

On the other hand, $\frac{1}{x} \to \pm\infty$ as $x \to 0$, and as $\frac{1}{x} \to \pm\infty$, $\sin\frac{1}{x}$ oscillates infinitely often. Thus

$$\lim_{x\to 0} \sin\frac{1}{x}$$

does not exist.

Exercises

1. What are the horizontal asymptotes of the function in Figure 6?

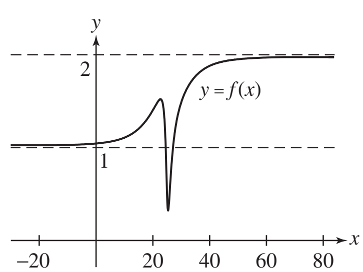


FIGURE 6

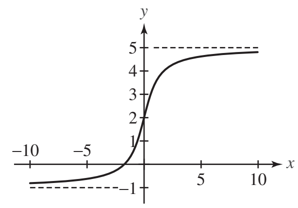
SOLUTION Because

$$\lim_{x\to-\infty} f(x) = 1 \quad \text{and} \quad \lim_{x\to\infty} f(x) = 2$$

the function f has horizontal asymptotes of $y = 1$ and $y = 2$.

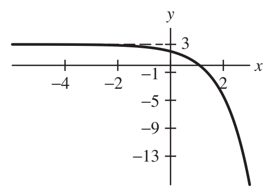
2. Sketch the graph of a function f that has both $y = -1$ and $y = 5$ as horizontal asymptotes.

SOLUTION



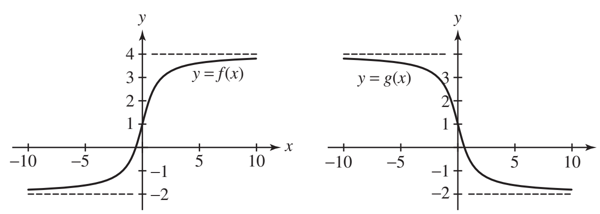
3. Sketch the graph of a function f with a single horizontal asymptote $y = 3$.

SOLUTION



4. Sketch the graphs of two functions f and g that have both $y = -2$ and $y = 4$ as horizontal asymptotes but $\lim_{x\to\infty} f(x) \neq \lim_{x\to\infty} g(x)$.

SOLUTION



5. GU Investigate the asymptotic behavior of $f(x) = \dfrac{x^2}{x^2+1}$ numerically and graphically:

(a) Make a table of values of $f(x)$ for $x = \pm 50, \pm 100, \pm 500, \pm 1000$.

(b) Plot the graph of f.

(c) What are the horizontal asymptotes of f?

SOLUTION

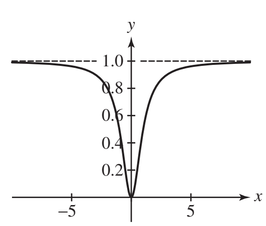
(a) From the table below, it appears that

$$\lim_{x\to\pm\infty} \frac{x^2}{x^2+1} = 1$$

x	± 50	± 100	± 500	± 1000
$f(x)$	0.999600	0.999900	0.999996	0.999999

(b) From the graph below, it also appears that

$$\lim_{x\to\pm\infty} \frac{x^2}{x^2+1} = 1$$



(c) The horizontal asymptote of f is $y = 1$.

6. GU Investigate $\lim_{x\to\pm\infty} \frac{12x+1}{\sqrt{4x^2+9}}$ numerically and graphically:

(a) Make a table of values of $f(x) = \frac{12x+1}{\sqrt{4x^2+9}}$ for $x = \pm 100, \pm 500, \pm 1000, \pm 10{,}000$.

(b) Plot the graph of f.

(c) What are the horizontal asymptotes of f?

SOLUTION

(a) From the tables below, it appears that

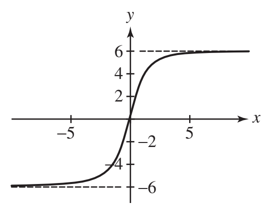
$$\lim_{x\to\infty} \frac{12x+1}{\sqrt{4x^2+9}} = 6 \quad \text{and} \quad \lim_{x\to-\infty} \frac{12x+1}{\sqrt{4x^2+9}} = -6$$

x	-100	-500	-1000	-10000
$f(x)$	-5.994326	-5.998973	-5.999493	-5.999950

x	100	500	1000	10000
$f(x)$	6.004325	6.000973	6.000493	6.000050

(b) From the graph below, it also appears that

$$\lim_{x\to\infty} \frac{12x+1}{\sqrt{4x^2+9}} = 6 \quad \text{and} \lim_{x\to-\infty} \frac{12x+1}{\sqrt{4x^2+9}} = -6$$



(c) The horizontal asymptotes of f are $y = -6$ and $y = 6$.

In Exercises 7–16, evaluate the limit.

7. $\lim_{x\to\infty} \frac{x}{x+9}$

SOLUTION

$$\lim_{x\to\infty} \frac{x}{x+9} = \lim_{x\to\infty} \frac{x^{-1}(x)}{x^{-1}(x+9)} = \lim_{x\to\infty} \frac{1}{1+\frac{9}{x}} = \frac{1}{1+0} = 1$$

8. $\lim_{x\to\infty} \frac{3x^2+20x}{4x^2+9}$

SOLUTION

$$\lim_{x\to\infty} \frac{3x^2+20x}{4x^2+9} = \lim_{x\to\infty} \frac{x^{-2}(3x^2+20x)}{x^{-2}(4x^2+9)} = \lim_{x\to\infty} \frac{3+\frac{20}{x}}{4+\frac{9}{x^2}} = \frac{3+0}{4+0} = \frac{3}{4}$$

9. $\lim_{x\to\infty} \frac{3x^2+20x}{2x^4+3x^3-29}$

SOLUTION

$$\lim_{x\to\infty} \frac{3x^2+20x}{2x^4+3x^3-29} = \lim_{x\to\infty} \frac{x^{-4}(3x^2+20x)}{x^{-4}(2x^4+3x^3-29)} = \lim_{x\to\infty} \frac{\frac{3}{x^2}+\frac{20}{x^3}}{2+\frac{3}{x}-\frac{29}{x^4}} = \frac{0}{2} = 0$$

10. $\lim_{x\to\infty} \frac{4}{x+5}$

SOLUTION

$$\lim_{x\to\infty} \frac{4}{x+5} = \lim_{x\to\infty} \frac{x^{-1}(4)}{x^{-1}(x+5)} = \lim_{x\to\infty} \frac{\frac{4}{x}}{1+\frac{5}{x}} = \frac{0}{1} = 0$$

11. $\lim_{x\to\infty} \frac{7x-9}{4x+3}$

SOLUTION

$$\lim_{x\to\infty} \frac{7x-9}{4x+3} = \lim_{x\to\infty} \frac{x^{-1}(7x-9)}{x^{-1}(4x+3)} = \lim_{x\to\infty} \frac{7-\frac{9}{x}}{4+\frac{3}{x}} = \frac{7}{4}$$

12. $\lim\limits_{x\to\infty} \dfrac{9x^2-2}{6-29x}$

SOLUTION

$$\lim_{x\to\infty} \frac{9x^2-2}{6-29x} = \lim_{x\to\infty} \frac{x^{-1}(9x^2-2)}{x^{-1}(6-29x)} = \lim_{x\to\infty} \frac{9x-\frac{2}{x}}{\frac{6}{x}-29} = -\infty$$

13. $\lim\limits_{x\to-\infty} \dfrac{7x^2-9}{4x+3}$

SOLUTION

$$\lim_{x\to-\infty} \frac{7x^2-9}{4x+3} = \lim_{x\to-\infty} \frac{x^{-1}(7x^2-9)}{x^{-1}(4x+3)} = \lim_{x\to-\infty} \frac{7x-\frac{9}{x}}{4+\frac{3}{x}} = -\infty$$

14. $\lim\limits_{x\to-\infty} \dfrac{5x-9}{4x^3+2x+7}$

SOLUTION

$$\lim_{x\to-\infty} \frac{5x-9}{4x^3+2x+7} = \lim_{x\to-\infty} \frac{x^{-3}(5x-9)}{x^{-3}(4x^3+2x+7)} = \lim_{x\to-\infty} \frac{\frac{5}{x^2}-\frac{9}{x^3}}{4+\frac{2}{x^2}+\frac{7}{x^3}} = \frac{0}{4} = 0$$

15. $\lim\limits_{x\to-\infty} \dfrac{3x^3-10}{x+4}$

SOLUTION

$$\lim_{x\to-\infty} \frac{3x^3-10}{x+4} = \lim_{x\to-\infty} \frac{x^{-1}(3x^3-10)}{x^{-1}(x+4)} = \lim_{x\to-\infty} \frac{3x^2-\frac{10}{x}}{1+\frac{4}{x}} = \infty$$

16. $\lim\limits_{x\to-\infty} \dfrac{2x^5+3x^4-31x}{8x^4-31x^2+12}$

SOLUTION

$$\lim_{x\to-\infty} \frac{2x^5+3x^4-31x}{8x^4-31x^2+12} = \lim_{x\to-\infty} \frac{x^{-4}(2x^5+3x^4-31x)}{x^{-4}(8x^4-31x^2+12)} = \lim_{x\to-\infty} \frac{2x+3-\frac{31}{x^3}}{8-\frac{31}{x^2}+\frac{12}{x^4}} = -\infty$$

In Exercises 17–24, find the horizontal asymptotes.

17. $f(x) = \dfrac{2x^2-3x}{8x^2+8}$

SOLUTION First calculate the limits as $x \to \pm\infty$. For $x \to \infty$,

$$\lim_{x\to\infty} \frac{2x^2-3x}{8x^2+8} = \lim_{x\to\infty} \frac{2-\frac{3}{x}}{8+\frac{8}{x^2}} = \frac{2}{8} = \frac{1}{4}$$

Similarly,

$$\lim_{x\to-\infty} \frac{2x^2-3x}{8x^2+8} = \lim_{x\to-\infty} \frac{2-\frac{3}{x}}{8+\frac{8}{x^2}} = \frac{2}{8} = \frac{1}{4}$$

Thus, the horizontal asymptote of f is $y = \frac{1}{4}$.

18. $f(x) = \dfrac{8x^3-x^2}{7+11x-4x^4}$

SOLUTION First calculate the limits as $x \to \pm\infty$. For $x \to \infty$,

$$\lim_{x\to\infty} \frac{8x^3 - x^2}{7 + 11x - 4x^4} = \lim_{x\to\infty} \frac{\frac{8}{x} - \frac{1}{x^2}}{\frac{7}{x^4} + \frac{11}{x^3} - 4} = 0$$

Similarly,

$$\lim_{x\to-\infty} \frac{8x^3 - x^2}{7 + 11x - 4x^4} = \lim_{x\to-\infty} \frac{\frac{8}{x} - \frac{1}{x^2}}{\frac{7}{x^4} + \frac{11}{x^3} - 4} = 0$$

Thus, the horizontal asymptote of f is $y = 0$.

19. $f(x) = \dfrac{\sqrt{36x^2 + 7}}{9x + 4}$

SOLUTION For $x > 0$, $x^{-1} = |x^{-1}| = \sqrt{x^{-2}}$, so

$$\lim_{x\to\infty} \frac{\sqrt{36x^2 + 7}}{9x + 4} = \lim_{x\to\infty} \frac{\sqrt{36 + \frac{7}{x^2}}}{9 + \frac{4}{x}} = \frac{\sqrt{36}}{9} = \frac{2}{3}$$

On the other hand, for $x < 0$, $x^{-1} = -|x^{-1}| = -\sqrt{x^{-2}}$, so

$$\lim_{x\to-\infty} \frac{\sqrt{36x^2 + 7}}{9x + 4} = \lim_{x\to-\infty} \frac{-\sqrt{36 + \frac{7}{x^2}}}{9 + \frac{4}{x}} = \frac{-\sqrt{36}}{9} = -\frac{2}{3}$$

Thus, the horizontal asymptotes of f are $y = \frac{2}{3}$ and $y = -\frac{2}{3}$.

20. $f(x) = \dfrac{\sqrt{36x^4 + 7}}{9x^2 + 4}$

SOLUTION For all $x \neq 0$, $x^{-2} = |x^{-2}| = \sqrt{x^{-4}}$, so

$$\lim_{x\to\infty} \frac{\sqrt{36x^4 + 7}}{9x^2 + 4} = \lim_{x\to\infty} \frac{\sqrt{36 + \frac{7}{x^4}}}{9 + \frac{4}{x^2}} = \frac{\sqrt{36}}{9} = \frac{2}{3}$$

Similarly,

$$\lim_{x\to-\infty} \frac{\sqrt{36x^4 + 7}}{9x^2 + 4} = \lim_{x\to-\infty} \frac{\sqrt{36 + \frac{7}{x^4}}}{9 + \frac{4}{x^2}} = \frac{\sqrt{36}}{9} = \frac{2}{3}$$

Thus, the horizontal asymptote of f is $y = \frac{2}{3}$.

21. $f(t) = \dfrac{3^t}{1 + 3^{-t}}$

SOLUTION With

$$\lim_{t\to\infty} \frac{3^t}{1 + 3^{-t}} = \infty$$

and

$$\lim_{t\to-\infty} \frac{3^t}{1 + 3^{-t}} = 0$$

the function f has one horizontal asymptote, $y = 0$.

22. $f(t) = \dfrac{t^{1/3}}{(64t^2 + 9)^{1/6}}$

SOLUTION For $t > 0$, $t^{-1/3} = |t^{-1/3}| = (t^{-2})^{1/6}$, so

$$\lim_{t\to\infty} \frac{t^{1/3}}{(64t^2 + 9)^{1/6}} = \lim_{t\to\infty} \frac{1}{(64 + \frac{9}{t^2})^{1/6}} = \frac{1}{2}$$

On the other hand, for $t < 0$, $t^{-1/3} = -|t^{-1/3}| = -(t^{-2})^{1/6}$, so

$$\lim_{t\to-\infty} \frac{t^{1/3}}{(64t^2 + 9)^{1/6}} = \lim_{t\to-\infty} \frac{1}{-(64 + \frac{9}{t^2})^{1/6}} = -\frac{1}{2}$$

Thus, the horizontal asymptotes for f are $y = \frac{1}{2}$ and $y = -\frac{1}{2}$.

23. $g(t) = \frac{10}{1+3^{-t}}$

SOLUTION Because

$$\lim_{t\to-\infty} 3^{-t} = \infty \quad \text{and} \quad \lim_{t\to\infty} 3^{-t} = 0$$

it follows that

$$\lim_{t\to-\infty} \frac{10}{1+3^{-t}} = 0 \quad \text{and} \quad \lim_{t\to-\infty} \frac{10}{1+3^{-t}} = 10$$

Thus, the horizontal asymptotes of g are $y = 0$ and $y = 10$.

24. $p(t) = 2^{-t^2}$

SOLUTION With

$$\lim_{t\to-\infty} 2^{-t^2} = 0 \quad \text{and} \quad \lim_{t\to\infty} 2^{-t^2} = 0$$

the function p has one horizontal asymptote, $y = 0$.

The following statement is incorrect: "If f has a horizontal asymptote $y = L$ at ∞, then the graph of f approaches the line $y = L$ as x gets greater and greater, but never touches it." In Exercises 25 and 26, determine $\lim_{x\to\infty} f(x)$ and indicate how f demonstrates that the statement is incorrect.

25. $f(x) = \frac{2x+|x|}{x}$

SOLUTION For $x > 0$, $|x| = x$ and

$$f(x) = \frac{2x + x}{x} = 3$$

Thus, $\lim_{x\to\infty} f(x) = 3$, so f has a horizontal asymptote of $y = 3$. The statement that the graph of f never touches this horizontal asymptote is incorrect because, for all $x > 0$, the graph of f coincides with the horizontal asymptote $y = 3$.

26. $f(x) = \frac{\sin x}{x}$

SOLUTION Because $-1 \le \sin x \le 1$, it follows that for $x \ne 0$,

$$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$$

With

$$\lim_{x\to\infty} \left(-\frac{1}{x}\right) = 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{1}{x} = 0$$

the Squeeze Theorem guarantees that

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\sin x}{x} = 0$$

Thus, f has a horizontal asymptote of $y = 0$. The statement that the graph of f never touches this horizontal asymptote is incorrect because the graph of f crosses $y = 0$ infinitely often (at $x = n\pi$ for every positive integer n, in particular) as $x \to \infty$.

In Exercises 27–34, evaluate the limit.

27. $\lim_{x\to\infty} \frac{\sqrt{9x^4 + 3x + 2}}{4x^3 + 1}$

SOLUTION For $x > 0$, $x^{-3} = |x^{-3}| = \sqrt{x^{-6}}$, so

$$\lim_{x\to\infty} \frac{\sqrt{9x^4 + 3x + 2}}{4x^3 + 1} = \lim_{x\to\infty} \frac{\sqrt{\frac{9}{x^2} + \frac{3}{x^5} + \frac{2}{x^6}}}{4 + \frac{1}{x^3}} = 0$$

28. $\lim_{x\to\infty} \frac{\sqrt{x^3 + 20x}}{10x - 2}$

SOLUTION For $x > 0$, $x^{-1} = |x^{-1}| = \sqrt{x^{-2}}$, so

$$\lim_{x\to\infty} \frac{\sqrt{x^3 + 20x}}{10x - 2} = \lim_{x\to\infty} \frac{\sqrt{x + \frac{20}{x}}}{10 - \frac{2}{x}} = \infty$$

29. $\lim_{x\to-\infty} \dfrac{8x^2+7x^{1/3}}{\sqrt{16x^4+6}}$

SOLUTION For $x<0$, $x^{-2}=|x^{-2}|=\sqrt{x^{-4}}$, so

$$\lim_{x\to-\infty} \frac{8x^2+7x^{1/3}}{\sqrt{16x^4+6}} = \lim_{x\to-\infty} \frac{8+\frac{7}{x^{5/3}}}{\sqrt{16+\frac{6}{x^4}}} = \frac{8}{\sqrt{16}} = 2$$

30. $\lim_{x\to-\infty} \dfrac{4x-3}{\sqrt{25x^2+4x}}$

SOLUTION For $x<0$, $x^{-1}=-|x^{-1}|=-\sqrt{x^{-2}}$, so

$$\lim_{x\to-\infty} \frac{4x-3}{\sqrt{25x^2+4x}} = \lim_{x\to-\infty} \frac{4-\frac{3}{x}}{-\sqrt{25+\frac{4}{x}}} = \frac{4}{-\sqrt{25}} = -\frac{4}{5}$$

31. $\lim_{t\to\infty} \dfrac{t^{4/3}+t^{1/3}}{(4t^{2/3}+1)^2}$

SOLUTION $\lim_{t\to\infty} \dfrac{t^{4/3}+t^{1/3}}{(4t^{2/3}+1)^2} = \lim_{t\to\infty} \dfrac{1+\frac{1}{t}}{(4+\frac{1}{t^{2/3}})^2} = \dfrac{1}{16}$

32. $\lim_{t\to\infty} \dfrac{t^{4/3}-9t^{1/3}}{(8t^4+2)^{1/3}}$

SOLUTION $\lim_{t\to\infty} \dfrac{t^{4/3}-9t^{1/3}}{(8t^4+2)^{1/3}} = \lim_{t\to\infty} \dfrac{1-\frac{9}{t}}{(8+\frac{2}{t^4})^{1/3}} = \dfrac{1}{2}$

33. $\lim_{x\to-\infty} \dfrac{|x|+x}{x+1}$

SOLUTION For $x<0$, $|x|=-x$. Therefore, for all $x<0$,

$$\frac{|x|+x}{x+1} = \frac{-x+x}{x+1} = 0$$

consequently,

$$\lim_{x\to-\infty} \frac{|x|+x}{x+1} = 0$$

34. $\lim_{t\to-\infty} \dfrac{4+6e^{2t}}{5-9e^{3t}}$

SOLUTION Because

$$\lim_{t\to-\infty} e^{2t} = \lim_{t\to-\infty} e^{3t} = 0$$

it follows that

$$\lim_{t\to-\infty} \frac{4+6e^{2t}}{5-9e^{3t}} = \frac{4+0}{5-0} = \frac{4}{5}$$

35. Determine $\lim_{t\to\infty} 5^{-1/t^2}$. Explain geometrically.

SOLUTION Because

$$\lim_{t\to\infty} \left(-\frac{1}{t^2}\right) = 0,$$

it follows that

$$\lim_{t\to\infty} 5^{-1/t^2} = 5^0 = 1.$$

Geometrically, this means that the graph of $y=5^{-1/t^2}$ has a horizontal asymptote at $y=1$.

36. Show that $\lim_{x\to\infty}(\sqrt{x^2+1}-x)=0$. *Hint:* Observe that

$$\sqrt{x^2+1}-x = \frac{1}{\sqrt{x^2+1}+x}$$

SOLUTION Rationalizing the "numerator," we find

$$\sqrt{x^2+1} - x = (\sqrt{x^2+1} - x)\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}$$
$$= \frac{(x^2+1)-x^2}{\sqrt{x^2+1}+x} = \frac{1}{\sqrt{x^2+1}+x}$$

Thus,

$$\lim_{x\to\infty}(\sqrt{x^2+1} - x) = \lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x} = 0$$

In Exercises 37–42, calculate the limit.

37. $\lim_{x\to\infty}(\sqrt{4x^4+9x} - 2x^2)$

SOLUTION Write

$$\sqrt{4x^4+9x} - 2x^2 = \left(\sqrt{4x^4+9x} - 2x^2\right)\frac{\sqrt{4x^4+9x}+2x^2}{\sqrt{4x^4+9x}+2x^2}$$
$$= \frac{(4x^4+9x)-4x^4}{\sqrt{4x^4+9x}+2x^2} = \frac{9x}{\sqrt{4x^4+9x}+2x^2}$$

Thus,

$$\lim_{x\to\infty}(\sqrt{4x^4+9x} - 2x^2) = \lim_{x\to\infty}\frac{9x}{\sqrt{4x^4+9x}+2x^2} = 0$$

38. $\lim_{x\to\infty}(\sqrt{9x^3+x} - x^{3/2})$

SOLUTION Write

$$\sqrt{9x^3+x} - x^{3/2} = \left(\sqrt{9x^3+x} - x^{3/2}\right)\frac{\sqrt{9x^3+x}+x^{3/2}}{\sqrt{9x^3+x}+x^{3/2}}$$
$$= \frac{(9x^3+x)-x^3}{\sqrt{9x^3+x}+x^{3/2}} = \frac{8x^3+x}{\sqrt{9x^3+x}+x^{3/2}}$$

Thus,

$$\lim_{x\to\infty}(\sqrt{9x^3+x} - x^{3/2}) = \lim_{x\to\infty}\frac{8x^3+x}{\sqrt{9x^3+x}+x^{3/2}} = \infty$$

39. $\lim_{x\to\infty}(2\sqrt{x} - \sqrt{x+2})$

SOLUTION Write

$$2\sqrt{x} - \sqrt{x+2} = (2\sqrt{x} - \sqrt{x+2})\frac{2\sqrt{x}+\sqrt{x+2}}{2\sqrt{x}+\sqrt{x+2}}$$
$$= \frac{4x-(x+2)}{2\sqrt{x}+\sqrt{x+2}} = \frac{3x-2}{2\sqrt{x}+\sqrt{x+2}}$$

Thus,

$$\lim_{x\to\infty}(2\sqrt{x} - \sqrt{x+2}) = \lim_{x\to\infty}\frac{3x-2}{2\sqrt{x}+\sqrt{x+2}} = \infty$$

40. $\lim_{x\to\infty}\left(\frac{1}{x} - \frac{1}{x+2}\right)$

SOLUTION Write

$$\frac{1}{x} - \frac{1}{x-2} = \frac{(x-2)-x}{x(x-2)} = \frac{-2}{x^2-2x}$$

Thus,

$$\lim_{x\to\infty}\left(\frac{1}{x} - \frac{1}{x-2}\right) = \lim_{x\to\infty}\frac{-2}{x^2-2x} = 0$$

41. $\lim_{x\to-\infty} \frac{|x| + x}{x + 1}$

SOLUTION For $x < 0$, $|x| = -x$. Therefore, for all $x < 0$,

$$\frac{|x| + x}{x + 1} = \frac{-x + x}{x + 1} = 0;$$

consequently,

$$\lim_{x\to-\infty} \frac{|x| + x}{x + 1} = 0.$$

42. $\lim_{t\to-\infty} \frac{4 + 5^{2t}}{5 - 5^{3t}}$

SOLUTION Because

$$\lim_{t\to-\infty} 5^{2t} = \lim_{t\to-\infty} 5^{3t} = 0,$$

it follows that

$$\lim_{t\to-\infty} \frac{4 + 5^{2t}}{5 - 5^{3t}} = \frac{4 + 0}{5 - 0} = \frac{4}{5}.$$

43. Let $P(n)$ be the perimeter of an n-gon inscribed in a unit circle (Figure 7).

(a) Explain, intuitively, why $P(n)$ approaches 2π as $n \to \infty$.

(b) Show that $P(n) = 2n \sin\left(\frac{\pi}{n}\right)$.

(c) Combine (a) and (b) to conclude that $\lim_{n\to\infty} \frac{n}{\pi} \sin\left(\frac{\pi}{n}\right) = 1$.

(d) Use this to give another argument that $\lim_{\theta\to0} \frac{\sin\theta}{\theta} = 1$.

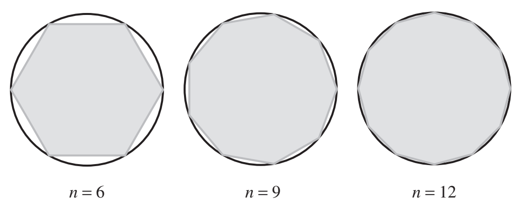


FIGURE 7

SOLUTION

(a) As $n \to \infty$, the n-gon approaches a circle of radius 1. Therefore, the perimeter of the n-gon approaches the circumference of the unit circle as $n \to \infty$. That is, $P(n) \to 2\pi$ as $n \to \infty$.

(b) Each side of the n-gon is the third side of an isosceles triangle with equal length sides of length 1 and angle $\theta = \frac{2\pi}{n}$ between the equal length sides. The length of each side of the n-gon is therefore

$$\sqrt{1^2 + 1^2 - 2\cos\frac{2\pi}{n}} = \sqrt{2\left(1 - \cos\frac{2\pi}{n}\right)} = \sqrt{4\sin^2\frac{\pi}{n}} = 2\sin\frac{\pi}{n}$$

Finally,

$$P(n) = 2n\sin\frac{\pi}{n}$$

(c) Combining parts (a) and (b),

$$\lim_{n\to\infty} P(n) = \lim_{n\to\infty} 2n\sin\frac{\pi}{n} = 2\pi$$

Dividing both sides of this last expression by 2π yields

$$\lim_{n\to\infty} \frac{n}{\pi}\sin\frac{\pi}{n} = 1$$

(d) Let $\theta = \frac{\pi}{n}$. Then $\theta \to 0$ as $n \to \infty$,

$$\frac{n}{\pi}\sin\frac{\pi}{n} = \frac{1}{\theta}\sin\theta = \frac{\sin\theta}{\theta}$$

and

$$\lim_{n\to\infty} \frac{n}{\pi}\sin\frac{\pi}{n} = \lim_{\theta\to0} \frac{\sin\theta}{\theta} = 1$$

44. Physicists have observed that Einstein's theory of **special relativity** reduces to Newtonian mechanics in the limit as $c \to \infty$, where c is the speed of light. This is illustrated by a stone tossed up vertically from ground level so that it returns to Earth 1 s later. Using Newton's Laws, we find that the stone's maximum height is $h = g/8$ m ($g = 9.8$ m/s^2). According to special relativity, the stone's mass depends on its velocity divided by c, and the maximum height is

$$h(c) = c\sqrt{c^2/g^2 + 1/4} - c^2/g$$

Prove that $\lim_{c\to\infty} h(c) = g/8$.

SOLUTION Write

$$h(c) = c\sqrt{c^2/g^2 + 1/4} - c^2/g = \left(c\sqrt{c^2/g^2 + 1/4} - c^2/g\right)\frac{c\sqrt{c^2/g^2 + 1/4} + c^2/g}{c\sqrt{c^2/g^2 + 1/4} + c^2/g}$$

$$= \frac{c^2(c^2/g^2 + 1/4) - c^4/g^2}{c\sqrt{c^2/g^2 + 1/4} + c^2/g} = \frac{c^2/4}{c\sqrt{c^2/g^2 + 1/4} + c^2/g}$$

Thus,

$$\lim_{c\to\infty} h(c) = \lim_{c\to\infty} \frac{c^2/4}{c\sqrt{c^2/g^2 + 1/4} + c^2/g} = \frac{c^2/4}{2c^2/g} = \frac{g}{8}$$

45. According to the **Michaelis–Menten equation**, when an enzyme is combined with a substrate of concentration s (in millimolars), the reaction rate (in micromolars/min) is

$$R(s) = \frac{As}{K + s} \qquad (A, K \text{ constants})$$

(a) Show, by computing $\lim_{s\to\infty} R(s)$, that A is the limiting reaction rate as the concentration s approaches ∞.

(b) Show that the reaction rate $R(s)$ attains one-half of the limiting value A when $s = K$.

(c) For a certain reaction, $K = 1.25$ mM and $A = 0.1$. For which concentration s is $R(s)$ equal to 75% of its limiting value?

SOLUTION

(a) $\lim_{s\to\infty} R(s) = \lim_{s\to\infty} \frac{As}{K + s} = \lim_{s\to\infty} \frac{A}{1 + \frac{K}{s}} = A$

(b) Observe that

$$R(K) = \frac{AK}{K + K} = \frac{AK}{2K} = \frac{A}{2}$$

half of the limiting value.

(c) By part (a), the limiting value is 0.1, so we need to determine the value of s that satisfies

$$R(s) = \frac{0.1s}{1.25 + s} = 0.075$$

Solving this equation for s yields

$$s = \frac{(1.25)(0.075)}{0.025} = 3.75 \text{ mM}$$

Further Insights and Challenges

46. Every limit as $x \to \infty$ can be rewritten as a one-sided limit as $t \to 0^+$, where $t = x^{-1}$. Setting $g(t) = f(t^{-1})$, we have

$$\lim_{x\to\infty} f(x) = \lim_{t\to 0^+} g(t)$$

Show that $\lim_{x\to\infty} \frac{3x^2 - x}{2x^2 + 5} = \lim_{t\to 0^+} \frac{3 - t}{2 + 5t^2}$, and evaluate using the Quotient Law.

SOLUTION Let $t = x^{-1}$. Then $x = t^{-1}$, $t \to 0+$ as $x \to \infty$, and

$$\frac{3x^2 - x}{2x^2 + 5} = \frac{3t^{-2} - t^{-1}}{2t^{-2} + 5} = \frac{3 - t}{2 + 5t^2}$$

Thus,

$$\lim_{x\to\infty} \frac{3x^2 - x}{2x^2 + 5} = \lim_{t\to 0+} \frac{3 - t}{2 + 5t^2} = \frac{3}{2}$$

47. Rewrite the following as one-sided limits as in Exercise 46 and evaluate.

(a) $\lim_{x\to\infty} \frac{3-12x^3}{4x^3+3x+1}$ **(b)** $\lim_{x\to\infty} 3^{1/x}$ **(c)** $\lim_{x\to\infty} x\sin\frac{1}{x}$

SOLUTION

(a) Let $t = x^{-1}$. Then $x = t^{-1}$, $t \to 0+$ as $x \to \infty$, and

$$\frac{3-12x^3}{4x^3+3x+1} = \frac{3-12t^{-3}}{4t^{-3}+3t^{-1}+1} = \frac{3t^3-12}{4+3t^2+t^3}$$

Thus,

$$\lim_{x\to\infty} \frac{3-12x^3}{4x^3+3x+1} = \lim_{t\to 0+} \frac{3t^3-12}{4+3t^2+t^3} = \frac{-12}{4} = -3$$

(b) Let $t = x^{-1}$. Then $x = t^{-1}$, $t \to 0+$ as $x \to \infty$, and $3^{1/x} = 3^t$. Thus,

$$\lim_{x\to\infty} 3^{1/x} = \lim_{t\to 0+} 3^t = 3^0 = 1$$

(c) Let $t = x^{-1}$. Then $x = t^{-1}$, $t \to 0+$ as $x \to \infty$, and

$$x\sin\frac{1}{x} = \frac{1}{t}\sin t = \frac{\sin t}{t}$$

Thus,

$$\lim_{x\to\infty} x\sin\frac{1}{x} = \lim_{t\to 0+} \frac{\sin t}{t} = 1$$

48. Let $G(b) = \lim_{x\to\infty}(1+b^x)^{1/x}$ for $b \ge 0$. Investigate $G(b)$ numerically and graphically for $b = 0.2, 0.8, 2, 3, 5$ (and additional values if necessary). Then make a conjecture for the value of $G(b)$ as a function of b. Draw a graph of $y = G(b)$. Does G appear to be continuous? We will evaluate $G(b)$ using L'Hôpital's Rule in Section 7.5 (see Exercise 69 there).

SOLUTION

- $b = 0.2$:

x	5	10	50	100
$f(x)$	1.000064	1.000000	1.000000	1.000000

  It appears that $G(0.2) = 1$.

- $b = 0.8$:

x	5	10	50	100
$f(x)$	1.058324	1.010251	1.000000	1.000000

  It appears that $G(0.8) = 1$.

- $b = 2$:

x	5	10	50	100
$f(x)$	2.012347	2.000195	2.000000	2.000000

  It appears that $G(2) = 2$.

- $b = 3$:

x	5	10	50	100
$f(x)$	3.002465	3.000005	3.000000	3.000000

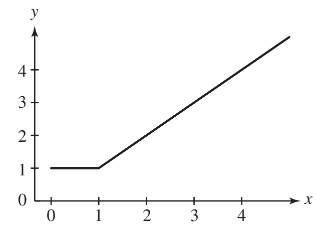
  It appears that $G(3) = 3$.

- $b = 5$:

x	5	10	50	100
$f(x)$	5.000320	5.000000	5.000000	5.000000

  It appears that $G(5) = 5$.

Based on these observations we conjecture that $G(b) = 1$ if $0 \le b \le 1$ and $G(b) = b$ for $b > 1$. The graph of $y = G(b)$ is shown below; the graph does appear to be continuous.



2.8 The Intermediate Value Theorem

Preliminary Questions

1. Prove that $f(x) = x^2$ takes on the value 0.5 in the interval $[0, 1]$.

SOLUTION Observe that $f(x) = x^2$ is continuous on $[0, 1]$ with $f(0) = 0$ and $f(1) = 1$. Because $f(0) < 0.5 < f(1)$, the Intermediate Value Theorem guarantees there is a $c \in [0, 1]$ such that $f(c) = 0.5$.

2. The temperature in Vancouver was 8°C at 6 AM and rose to 20°C at noon. Which assumption about temperature allows us to conclude that the temperature was 15°C at some moment of time between 6 AM and noon?

SOLUTION We must assume that temperature is a continuous function of time.

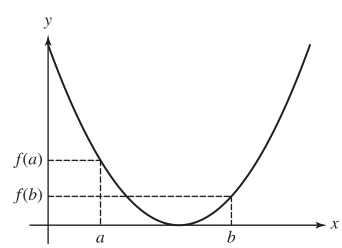
3. What is the graphical interpretation of the IVT?

SOLUTION If f is continuous on $[a, b]$, then the horizontal line $y = k$ for every k between $f(a)$ and $f(b)$ intersects the graph of $y = f(x)$ at least once.

4. Show that the following statement is false by drawing a graph that provides a counterexample:

If f is continuous and has a root in $[a, b]$, then $f(a)$ and $f(b)$ have opposite signs.

SOLUTION



5. Assume that f is continuous on $[1, 5]$ and that $f(1) = 20$, $f(5) = 100$. Determine whether each of the following statements is always true, never true, or sometimes true.

(a) $f(c) = 3$ has a solution with $c \in [1, 5]$.

(b) $f(c) = 75$ has a solution with $c \in [1, 5]$.

(c) $f(c) = 50$ has no solution with $c \in [1, 5]$.

(d) $f(c) = 30$ has exactly one solution with $c \in [1, 5]$.

SOLUTION

(a) This statement is sometimes true. Because 3 does not lie between 20 and 100, the IVT cannot be used to guarantee that the function takes on the value 3 but it may still do so.

(b) This statement is always true. Because f is continuous on $[1, 5]$ and $20 = f(1) < 75 < f(5) = 100$, the IVT guarantees there exists a $c \in [1, 5]$ such that $f(c) = 75$.

(c) This statement is never true. Because f is continuous on $[1, 5]$ and $20 = f(1) < 50 < f(5) = 100$, the IVT guarantees there exists a $c \in [1, 5]$ such that $f(c) = 50$.

(d) This statement is sometimes true. Because f is continuous on $[1, 5]$ and $20 = f(1) < 30 < f(5) = 100$, the IVT guarantees there exists a $c \in [1, 5]$ such that $f(c) = 30$ but there may be more than one such value for c.

Exercises

1. Use the IVT to show that $f(x) = x^3 + x$ takes on the value 9 for some x in $[1, 2]$.

SOLUTION Observe that $f(1) = 2$ and $f(2) = 10$. Since f is a polynomial, it is continuous everywhere; in particular on $[1, 2]$. Therefore, by the IVT there is a $c \in [1, 2]$ such that $f(c) = 9$.

2. Show that $g(t) = \dfrac{t}{t+1}$ takes on the value 0.499 for some t in $[0, 1]$.

SOLUTION $g(0) = 0$ and $g(1) = \frac{1}{2}$. Since $g(t)$ is continuous for all $x \neq -1$, and since $0 < .499 < \frac{1}{2}$, the IVT states that $g(t) = .499$ for some t between 0 and 1.

3. Show that $g(t) = t^2 \tan t$ takes on the value $\frac{1}{2}$ for some t in $[0, \frac{\pi}{4}]$.

SOLUTION $g(0) = 0$ and $g(\frac{\pi}{4}) = \frac{\pi^2}{16}$. $g(t)$ is continuous for all t between 0 and $\frac{\pi}{4}$, and $0 < \frac{1}{2} < \frac{\pi^2}{16}$; therefore, by the IVT, there is a $c \in [0, \frac{\pi}{4}]$ such that $g(c) = \frac{1}{2}$.

4. Show that $f(x) = \frac{x^2}{x^7+1}$ takes on the value 0.4.

SOLUTION $f(0) = 0 < .4$. $f(1) = \frac{1}{2} > .4$. $f(x)$ is continuous at all points x where $x \neq -1$, therefore $f(x) = .4$ for some x between 0 and 1.

5. Show that $\cos x = x$ has a solution in the interval $[0, 1]$. *Hint:* Show that $f(x) = x - \cos x$ has a zero in $[0, 1]$.

SOLUTION Let $f(x) = x - \cos x$. Observe that f is continuous with $f(0) = -1$ and $f(1) = 1 - \cos 1 \approx .46$. Therefore, by the IVT there is a $c \in [0, 1]$ such that $f(c) = c - \cos c = 0$. Thus $c = \cos c$ and hence the equation $\cos x = x$ has a solution c in $[0, 1]$.

6. Use the IVT to find an interval of length $\frac{1}{2}$ containing a root of $f(x) = x^3 + 2x + 1$.

SOLUTION Let $f(x) = x^3 + 2x + 1$. Observe that $f(-1) = -2$ and $f(0) = 1$. Since f is continuous, we may conclude by the IVT that f has a root in $[-1, 0]$. Now, $f(-\frac{1}{2}) = -\frac{1}{8}$ so $f(-\frac{1}{2})$ and $f(0)$ are of opposite sign. Therefore, the IVT guarantees that f has a root on $[-\frac{1}{2}, 0]$.

In Exercises 7–16, prove using the IVT.

7. $\sqrt{c} + \sqrt{c+2} = 3$ has a solution.

SOLUTION Let $f(x) = \sqrt{x} + \sqrt{x+2} - 3$. Note that f is continuous on $[0, 2]$ with $f(0) = \sqrt{0} + \sqrt{2} - 3 \approx -1.59$ and $f(2) = \sqrt{2} + \sqrt{4} - 3 \approx 0.41$. Therefore, by the IVT there is a $c \in [0, 2]$ such that $f(c) = \sqrt{c} + \sqrt{c+2} - 3 = 0$. Thus $\sqrt{c} + \sqrt{c+2} = 3$, and the equation $\sqrt{c} + \sqrt{c+2} = 3$ has a solution c in $[0, 2]$.

8. For all integers n, $\sin nx = \cos x$ for some $x \in [0, \pi]$.

SOLUTION For each integer n, let $f(x) = \sin nx - \cos x$. Observe that f is continuous with $f(0) = -1$ and $f(\pi) = 1$. Therefore, by the IVT there is a $c \in [0, \pi]$ such that $f(c) = \sin nc - \cos c = 0$. Thus $\sin nc = \cos c$ and hence the equation $\sin nx = \cos x$ has a solution c in the interval $[0, \pi]$.

9. $\sqrt{2}$ exists. *Hint:* Consider $f(x) = x^2$.

SOLUTION Let $f(x) = x^2$. Observe that f is continuous with $f(1) = 1$ and $f(2) = 4$. Therefore, by the IVT there is a $c \in [1, 2]$ such that $f(c) = c^2 = 2$. This proves the existence of $\sqrt{2}$, a number whose square is 2.

10. A positive number c has an nth root for all positive integers n.

SOLUTION If $c = 1$, then $\sqrt[n]{c} = 1$. Now, suppose $c \neq 1$. Let $f(x) = x^n - c$, and let $b = \max\{1, c\}$. Then, if $c > 1$, $b^n = c^n > c$, and if $c < 1$, $b^n = 1 > c$. So $b^n > c$. Now observe that $f(0) = -c < 0$ and $f(b) = b^n - c > 0$. Since f is continuous on $[0, b]$, by the Intermediate Value Theorem, there is some $d \in [0, b]$ such that $f(d) = 0$. We can refer to d as $\sqrt[n]{c}$.

11. For all positive integers k, $\cos x = x^k$ has a solution.

SOLUTION For each positive integer k, let $f(x) = x^k - \cos x$. Observe that f is continuous on $\left[0, \frac{\pi}{2}\right]$ with $f(0) = -1$ and $f(\frac{\pi}{2}) = \left(\frac{\pi}{2}\right)^k > 0$. Therefore, by the IVT there is a $c \in \left[0, \frac{\pi}{2}\right]$ such that $f(c) = c^k - \cos(c) = 0$. Thus $\cos c = c^k$ and hence the equation $\cos x = x^k$ has a solution c in the interval $\left[0, \frac{\pi}{2}\right]$.

12. $2^x = bx$ has a solution if $b > 2$.

SOLUTION Let $f(x) = 2^x - bx$. Observe that f is continuous on $[0, 1]$ with $f(0) = 1 > 0$ and $f(1) = 2 - b < 0$ provided $b > 2$. Therefore, by the IVT, there is a $c \in [0, 1]$ such that $f(c) = 2^c - bc = 0$, provided $b > 2$. Hence, the equation $2^x = bx$ has a solution if $b > 2$.

13. $2^x + 3^x = 4^x$ has a solution.

SOLUTION Let $f(x) = 2^x + 3^x - 4^x$. Observe that f is continuous on $[0, 2]$ with $f(0) = 2^0 + 3^0 - 4^0 = 1 + 1 - 1 = 1$ and $f(2) = 2^2 + 3^2 - 4^2 = 4 + 9 - 16 = -3$. Therefore, by the IVT, there is a $c \in [0, 2]$ such that $f(c) = 2^c + 3^c - 4^c = 0$. Hence, the equation $2^x + 3^x = 4^x$ has a solution.

14. $\cos x = \tan 2x$ has a solution in $(0, 1)$.

SOLUTION Let $f(x) = \cos x - \tan 2x$. Observe that f is continuous on $[0, b]$ for any $0 < b < \pi/4 \approx 0.785$. In particular, f is continuous on $[0, 0.7]$. Now, $f(0) = \cos 0 - \tan 0 = 1$, and $f(0.7) = \cos 0.7 - \tan 1.4 \approx -5.03$. Therefore, by the IVT, there is a $c \in [0, 0.7]$ such that $f(c) = \cos c - \tan 2c = 0$. Because we know that 0 is not a solution, it follows that the equation $\cos x = \tan 2x$ has a solution in $[0, 0.7]$, which is contained in $(0, 1)$.

15. $2^x + \dfrac{1}{x} = -4$ has a solution.

SOLUTION Let $f(x) = 2^x + \dfrac{1}{x} + 4$. Observe that f is continuous for $x < 0$ with $f(-1) = 2^{-1} + \frac{1}{-1} + 4 = \frac{7}{2} > 0$ and $f\left(-\frac{1}{8}\right) = 2^{-1/8} - 8 + 4 \approx -3.08 < 0$. Therefore, by the IVT, there is a $c \in \left(-1, -\frac{1}{8}\right)$ such that $f(c) = 2^c - \frac{1}{c} + 4 = 0$ and thus $2^c - \frac{1}{c} = -4$.

16. $x^{1/3} = 1/(x-1)$ has a solution in $(1, 2)$.

SOLUTION Let $f(x) = x^{1/3} - 1/(x-1)$. Observe that f is continuous on $[b, 2]$ for an y $1 < b < 2$. In particular, f is continuous on $[1.1, 2]$. Now $f(1.1) = 1.1^{1/3} - 1/(1.1-1) \approx -8.97$ and $f(2) = 2^{1/3} - 1/(2-1) \approx 0.26$. Therefore, by the IVT, there is a $c \in [1.1, 2]$ such that $f(c) = c^{1/3} - 1/(c-1) = 0$. Because we know that 2 is not a solution, it follows that the equation $x^{1/3} = 1/(x-1)$ has a solution in $[1.1, 2)$, which is contained in $(1, 2)$.

17. Use the Intermediate Value Theorem to show that the equation $x^6 - 8x^4 + 10x^2 - 1 = 0$ has at least six distinct solutions.

SOLUTION Let $f(x) = x^6 - 8x^4 + 10x^2 - 1$. Then $f(0) = -1$, $f(\pm 1) = 2$, $f(\pm 2) = -25$ and $f(\pm 3) = 170$. Hence as we move along the number line from left to right through the points $-3, -2, -1, 0, 1, 2, 3$, the function changes sign at least 6 times. Hence there must be a zero of the function between any two of these integers, and therefore, there must be at least six distinct solutions to the equation $x^6 - 8x^4 + 10x^2 - 1 = 0$.

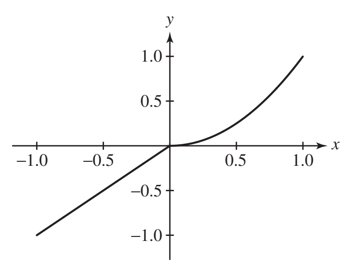
In Exercises 18–20, determine whether or not the IVT applies to show that the given function takes on all values between $f(a)$ and $f(b)$ for $x \in (a, b)$. If it does not apply, determine any values between $f(a)$ and $f(b)$ that the function does not take on for $x \in (a, b)$.

18.

$$f(x) = \begin{cases} x & \text{for } x < 0 \\ x^2 & \text{for } x \geq 0 \end{cases}$$

for the interval $[-1, 1]$.

SOLUTION The graph of f over the interval $[-1, 1]$ is shown below. From the graph, we see that f is continuous on $[-1, 1]$, so the IVT applies to show that this function takes on all values between $f(-1) = -1$ and $f(1) = 1$ for $x \in (-1, 1)$.

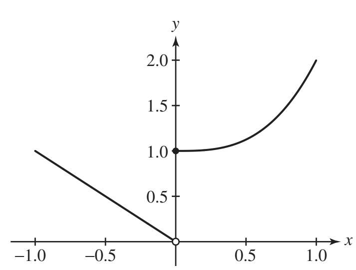


19.

$$f(x) = \begin{cases} -x & \text{for } x < 0 \\ x^3 + 1 & \text{for } x \geq 0 \end{cases}$$

for the interval $[-1, 1]$.

SOLUTION The graph of f over the interval $[-1, 1]$ is shown below. From the graph, we see that f is not continuous on $[-1, 1]$ because of a jump discontinuity at $x = 0$. Therefore, the IVT does not apply. However, from the graph, we see that f does take on every value between $f(-1) = 1$ and $f(1) = 2$ for $x \in (-1, 1)$.

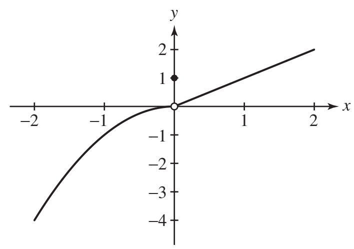


20.

$$f(x) = \begin{cases} -x^2 & \text{for } x < 0 \\ 1 & \text{for } x = 0 \\ x & \text{for } x > 0 \end{cases}$$

for the interval $[-2, 2]$.

SOLUTION The graph of f over the interval $[-2, 2]$ is shown below. From the graph, we see that f is not continuous on $[-1, 1]$ because of a removable discontinuity at $x = 0$. Therefore, the IVT does not apply. Moreover, from the graph, we see that the only value between $f(-2) = -4$ and $f(2) = 2$ that f does not take on for $x \in (-1, 1)$ is $y = 0$.



21. Carry out three steps of the Bisection Method for $f(x) = 2^x - x^3$ as follows:

(a) Show that f has a zero in $[1, 1.5]$.

(b) Show that f has a zero in $[1.25, 1.5]$.

(c) Determine whether $[1.25, 1.375]$ or $[1.375, 1.5]$ contains a zero.

SOLUTION Note that $f(x)$ is continuous for all x.

(a) $f(1) = 1$, $f(1.5) = 2^{1.5} - (1.5)^3 < 3 - 3.375 < 0$. Hence, $f(x) = 0$ for some x between 1 and 1.5.

(b) $f(1.25) \approx 0.4253 > 0$ and $f(1.5) < 0$. Hence, $f(x) = 0$ for some x between 1.25 and 1.5.

(c) $f(1.375) \approx -0.0059$. Hence, $f(x) = 0$ for some x between 1.25 and 1.375.

22. Figure 6 shows that $f(x) = x^3 - 8x - 1$ has a root in the interval $[2.75, 3]$. Apply the Bisection Method twice to find an interval of length $\frac{1}{16}$ containing this root.

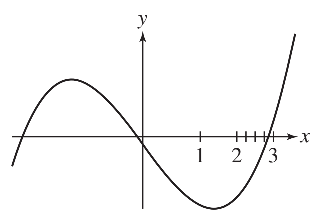


FIGURE 6 Graph of $y = x^3 - 8x - 1$.

SOLUTION Let $f(x) = x^3 - 8x - 1$. Observe that f is continuous with $f(2.75) = -2.203125$ and $f(3) = 2$. Therefore, by the IVT there is a $c \in [2.75, 3]$ such that $f(c) = 0$. The midpoint of the interval $[2.75, 3]$ is 2.875 and $f(2.875) = -0.236$. Hence, $f(x) = 0$ for some x between 2.875 and 3. The midpoint of the interval $[2.875, 3]$ is 2.9375 and $f(2.9375) = 0.84$. Thus, $f(x) = 0$ for some x between 2.875 and 2.9375.

23. Find an interval of length $\frac{1}{4}$ in $[1, 2]$ containing a root of the equation $x^7 + 3x - 10 = 0$.

SOLUTION Let $f(x) = x^7 + 3x - 10$. Observe that f is continuous on $[1, 2]$ with $f(1) = -6$ and $f(2) = 124$, so the IVT guarantees that the equation $x^7 + 3x - 10 = 0$ has a root on the interval $[1, 2]$. The midpoint of the interval $[1, 2]$ is 1.5 and $f(1.5) = 11.585938 > 0$, so we can conclude that the equation $x^7 + 3x - 10 = 0$ has a root on the interval $[1, 1.5]$. Finally, the midpoint of the interval $[1, 1.5]$ is 1.25 and $f(1.25) = -1.481628 < 0$, so we can conclude that the equation $x^7 + 3x - 10 = 0$ has a root on the interval $[1.25, 1.5]$.

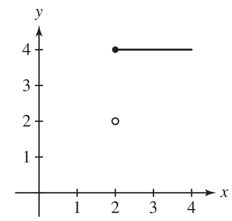
24. Show that $\tan^3\theta - 8\tan^2\theta + 17\tan\theta - 8 = 0$ has a root in $[0.5, 0.6]$. Apply the Bisection Method twice to find an interval of length 0.025 containing this root.

SOLUTION Let $f(x) = \tan^3\theta - 8\tan^2\theta + 17\tan\theta - 8$. Since $f(.5) = -.937387 < 0$ and $f(.6) = 0.206186 > 0$, we conclude that $f(x) = 0$ has a root in $[0.5, 0.6]$. Since $f(.55) = -0.35393 < 0$ and $f(.6) > 0$, we can conclude that $f(x) = 0$ has a root in $[0.55, 0.6]$. Since $f(.575) = -0.0707752 < 0$, we can conclude that f has a root on $[0.575, 0.6]$.

In Exercises 25–28, draw the graph of a function f on $[0, 4]$ with the given property.

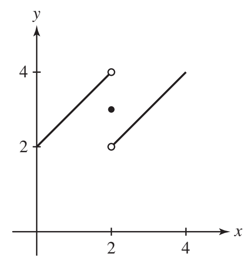
25. Jump discontinuity at $x = 2$ and does not satisfy the conclusion of the IVT

SOLUTION The function graphed below has a jump discontinuity at $x = 2$. Note that while $f(0) = 2$ and $f(4) = 4$, there is no point c in the interval $[0, 4]$ such that $f(c) = 3$. Accordingly, the conclusion of the IVT is *not* satisfied.



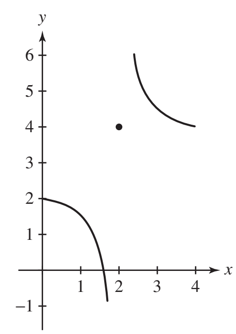
26. Jump discontinuity at $x = 2$ and satisfies the conclusion of the IVT on $[0, 4]$

SOLUTION The function graphed below has a jump discontinuity at $x = 2$. Note that for every value M between $f(0) = 2$ and $f(4) = 4$, there *is* a point c in the interval $[0, 4]$ such that $f(c) = M$. Accordingly, the conclusion of the IVT *is* satisfied.



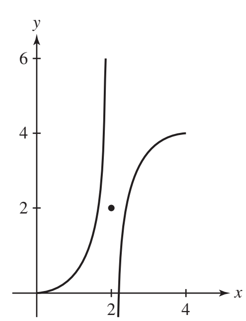
27. Infinite one-sided limits at $x = 2$ and does not satisfy the conclusion of the IVT

SOLUTION The function graphed below has infinite one-sided limits at $x = 2$. Note that while $f(0) = 2$ and $f(4) = 4$, there is no point c in the interval $[0, 4]$ such that $f(c) = 3$. Accordingly, the conclusion of the IVT is *not* satisfied.



28. Infinite one-sided limits at $x = 2$ and satisfies the conclusion of the IVT on $[0, 4]$.

SOLUTION The function graphed below has infinite one-sided limits at $x = 2$. Note that for every value M between $f(0) = 0$ and $f(4) = 4$, there *is* a point c in the interval $[0, 4]$ such that $f(c) = M$. Accordingly, the conclusion of the IVT *is* satisfied.



29. Can Corollary 2 be applied to $f(x) = x^{-1}$ on $[-1, 1]$? Does f have any roots?

SOLUTION Although $f(-1) = -1 < 0$ and $f(1) = 1 > 0$ are of opposite sign, Corollary 2 cannot be applied because f is not continuous on the interval $[-1, 1]$. This function does not have any roots.

30. **(a)** Assume that g and h are continuous on $[a, b]$. Use Corollary 2 to show that if $g(a) < h(a)$ and $h(b) < g(b)$, then there exists $c \in [a, b]$ such that $g(c) = h(c)$.

(b) Interpret the result of (a) in terms of the graphs of g and h, and show, by a graphical example, that the conclusion in (a) need not hold if one of g or h is not continuous.

SOLUTION

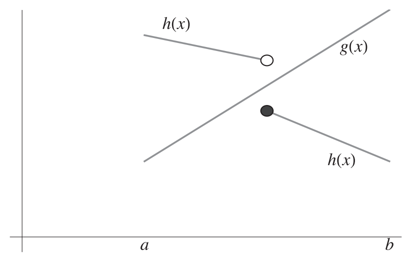
(a) Define the function f on $[a, b]$ by $f(x) = g(x) - h(x)$. Because f is the difference of two functions that are continuous on $[a, b]$, f is also continuous on $[a, b]$. Now,

$$f(a) = g(a) - h(a) < 0 \quad \text{and} \quad f(b) = g(b) - h(b) > 0.$$

Corollary 2 then implies there exists $c \in [a, b]$ such that $f(c) = 0$. It follows that at $x = c$, $g(c) = h(c)$.

(b) The result in (a) indicates that if g and h are continuous on $[a, b]$ with the graph of g below the graph of h at $x = a$ and with the graph of h below the graph of g at $x = b$, then there must exist a place in $[a, b]$ where the graphs intersect.

In the figure below, the functions g and h satisfy the assumptions in part (a), except that h is not continuous on $[a, b]$. The conclusion in (a) does not hold because the graphs of g and h do not intersect.



31. At 1:00 PM Jacqueline began to climb up Waterpail Hill from the bottom. At the same time Giles began to climb down from the top. Giles reached the bottom at 2:20 PM, when Jacqueline was 85% of the way up. Jacqueline reached the top at 2:50. Use the result in Exercise 30 to prove that there was a time when they were at the same elevation on the hill.

SOLUTION Let t represent time in minutes since 1:00 PM, and let $J(t)$ and $G(t)$ represent the elevation in percentage of the way up the hill at time t of Jacqueline and Giles, respectively. Then $J(0) = 0 < 1 = G(0)$ and $G(80) = 0 < 0.85 = J(80)$. Assuming that J and G are continuous on $[0, 80]$, the result in Exercise 30 implies there exists $c \in [0, 80]$ such that $J(c) = G(c)$. At time c, Jacqueline and Giles were at the same elevation on the hill.

32. On Wednesday at noon the weather was fair in Boston with a barometric pressure of 1018 mb. At the same time, a low-pressure storm system was passing by Buffalo, where the pressure was 996 mb. At noon Thursday the storm was approaching Boston, where the pressure was 1002 mb, while the weather was clearing in Buffalo and the pressure there had risen to 1014 mb. Use the result in Exercise 30 to prove that there was a time between noon Wednesday and noon Thursday when Boston and Buffalo had the same barometric pressure.

SOLUTION Let t represent time in hours since noon on Wednesday, and let $g(t)$ and $h(t)$ represent the barometric pressure in mb at time t in Boston and Buffalo, respectively. Then $h(0) = 996 < 1018 = g(0)$ and $g(24) = 1002 < 1014 = h(24)$. Assuming that g and h are continuous on $[0, 24]$, the result in Exercise 30 implies there exists $c \in [0, 24]$ such that $g(c) = h(c)$. At time c between noon on Wednesday and noon on Thursday, Boston and Buffalo had the same barometric pressure.

Further Insights and Challenges

Exercises 33 and 34 address the 1-Dimensional Brouwer Fixed Point Theorem. It indicates that every continuous function f mapping the closed interval $[0, 1]$ to itself must have a fixed point; that is, a point c such that $f(c) = c$.

33. Show that if f is continuous and $0 \le f(x) \le 1$ for $0 \le x \le 1$, then $f(c) = c$ for some c in $[0, 1]$ (Figure 7).

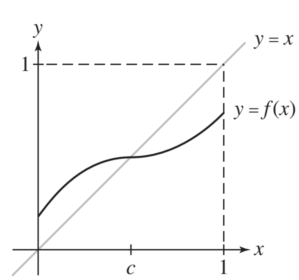


FIGURE 7 A function satisfying $0 \le f(x) \le 1$ for $0 \le x \le 1$.

SOLUTION If $f(0) = 0$, the proof is done with $c = 0$. We may assume that $f(0) > 0$. Let $g(x) = f(x) - x$. $g(0) = f(0) - 0 = f(0) > 0$. Since $f(x)$ is continuous, the Rule of Differences dictates that $g(x)$ is continuous. We need to prove that $g(c) = 0$ for some $c \in [0, 1]$. Since $f(1) \le 1$, $g(1) = f(1) - 1 \le 0$. If $g(1) = 0$, the proof is done with $c = 1$, so let's assume that $g(1) < 0$.

We now have a continuous function $g(x)$ on the interval $[0, 1]$ such that $g(0) > 0$ and $g(1) < 0$. From the IVT, there must be some $c \in [0, 1]$ so that $g(c) = 0$, so $f(c) - c = 0$ and so $f(c) = c$.

34. **(a)** Give an example showing that if f is continuous and $0 < f(x) < 1$ for $0 < x < 1$, then there does not need to be a c in $(0, 1)$ such that $f(c) = c$.

(b) Give an example showing that if $0 \le f(x) \le 1$ for $0 \le x \le 1$, but f is not necessarily continuous, then there does not need to be a c in $(0, 1)$ such that $f(c) = c$.

SOLUTION

(a) Let $f(x) = x/2$. For $0 < x < 1$, f is continuous and satisfies the condition that $0 < f(x) < 1$. Now, the equation $f(c) = c$, or equivalently $c/2 = c$, has as its only solution $c = 0$, which does not lie in the interval $(0, 1)$. Thus, there does not exist a $c \in (0, 1)$ such that $f(c) = c$.

(b) Let

$$f(x) = \begin{cases} 1 & \text{when } 0 \le x < \frac{1}{2} \\ 0 & \text{when } \frac{1}{2} \le x \ge 1 \end{cases}$$

For $0 \le x \le 1$, this function satisfies the condition that $0 \le f(x) \le 1$ but is not continuous on $[0, 1]$ because of a discontinuity at $x = \frac{1}{2}$. For this function, the equation $f(c) = c$ has no solution. Thus, there does not exist a $c \in (0, 1)$ such that $f(c) = c$.

35. Use the IVT to show that if f is continuous and one-to-one on an interval $[a, b]$, then f is either an increasing or a decreasing function.

SOLUTION Let $f(x)$ be a continuous, one-to-one function on the interval $[a, b]$. Suppose for sake of contradiction that $f(x)$ is neither increasing nor decreasing on $[a, b]$. Now, $f(x)$ cannot be constant, for that would contradict the condition that $f(x)$ is one-to-one. It follows that somewhere on $[a, b]$, $f(x)$ must transition from increasing to decreasing or from decreasing to increasing. To be specific, suppose $f(x)$ is increasing for $x_1 < x < x_2$ and decreasing for $x_2 < x < x_3$. Let k be any number between $\max\{f(x_1), f(x_3)\}$ and $f(x_2)$. Because $f(x)$ is continuous, the IVT guarantees there exists a $c_1 \in (x_1, x_2)$ such that $f(c_1) = k$; moreover, there exists a $c_2 \in (x_2, x_3)$ such that $f(c_2) = k$. However, this contradicts the condition that $f(x)$ is one-to-one. A similar analysis for the case when $f(x)$ is decreasing for $x_1 < x < x_2$ and increasing for $x_2 < x < x_3$ again leads to a contradiction. Therefore, $f(x)$ must be either increasing or decreasing on $[a, b]$.

36. **Ham Sandwich Theorem** Figure 8(A) shows a slice of ham. Prove that for any angle θ ($0 \le \theta \le \pi$), it is possible to cut the slice in half with a cut of incline θ. *Hint:* The lines of inclination θ are given by the equations $y = (\tan\theta)x + b$, where b varies from $-\infty$ to ∞. Each such line divides the slice into two pieces (one of which may be empty). Let $A(b)$ be the amount of ham to the left of the line minus the amount to the right, and let A be the total area of the ham. Show that $A(b) = -A$ if b is sufficiently large and $A(b) = A$ if b is sufficiently negative. Then use the IVT. This works if $\theta \ne 0$ or $\frac{\pi}{2}$. If $\theta = 0$, define $A(b)$ as the amount of ham above the line $y = b$ minus the amount below. How can you modify the argument to work when $\theta = \frac{\pi}{2}$ (in which case $\tan\theta = \infty$)?

SOLUTION Let θ be such that $\theta \ne \frac{\pi}{2}$. For any b, consider the line $L(\theta)$ drawn at angle θ to the x axis starting at $(0, b)$. This line has formula $y = (\tan\theta)x + b$. Let $A(b)$ be the amount of ham above the line minus that below the line.

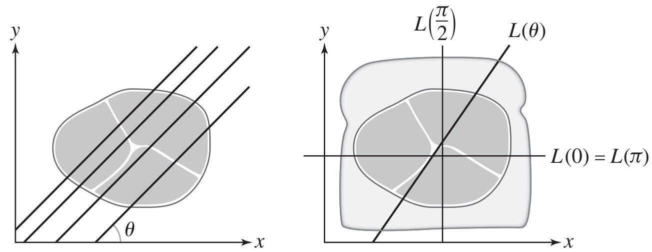
Let $A > 0$ be the area of the ham. We have to accept the following (reasonable) assumptions:

- For low enough $b = b_0$, the line $L(\theta)$ lies entirely below the ham, so that $A(b_0) = A - 0 = A$.
- For high enough b_1, the line $L(\theta)$ lies entirely above the ham, so that $A(b_1) = 0 - A = -A$.
- $A(b)$ is continuous as a function of b.

Under these assumptions, we see $A(b)$ is a continuous function satisfying $A(b_0) > 0$ and $A(b_1) < 0$ for some $b_0 < b_1$. By the IVT, $A(b) = 0$ for some $b \in [b_0, b_1]$.

Suppose that $\theta = \frac{\pi}{2}$. Let the line $L(c)$ be the vertical line through $(c, 0)$ ($x = c$). Let $A(c)$ be the area of ham to the left of $L(c)$ minus that to the right of $L(c)$. Since $L(0)$ lies entirely to the left of the ham, $A(0) = 0 - A = -A$. For some $c = c_1$ sufficiently large, $L(c)$ lies entirely to the right of the ham, so that $A(c_1) = A - 0 = A$. Hence $A(c)$ is a continuous function of c such that $A(0) < 0$ and $A(c_1) > 0$. By the IVT, there is some $c \in [0, c_1]$ such that $A(c) = 0$.

37. Figure 8(B) shows a slice of ham on a piece of bread. Prove that it is possible to slice this open-faced sandwich so that each part has equal amounts of ham and bread. *Hint:* By Exercise 36, for all $0 \le \theta \le \pi$ there is a line $L(\theta)$ of incline θ (which we assume is unique) that divides the ham into two equal pieces. Let $B(\theta)$ denote the amount of bread to the left of (or above) $L(\theta)$ minus the amount to the right (or below). Notice that $L(\pi)$ and $L(0)$ are the same line, but $B(\pi) = -B(0)$ since left and right get interchanged as the angle moves from 0 to π. Assume that B is continuous and apply the IVT. (By a further extension of this argument, one can prove the full Ham Sandwich Theorem, which states that if you allow the knife to cut at a slant, then it is possible to cut a sandwich consisting of a slice of ham and two slices of bread so that all three layers are divided in half.)



(A) Cutting a slice of ham at an angle θ

(B) A slice of ham on top of a slice of bread

FIGURE 8

SOLUTION For each angle θ, $0 \le \theta < \pi$, let $L(\theta)$ be the line at angle θ to the x-axis that slices the ham exactly in half, as shown in Figure 8. Let $L(0) = L(\pi)$ be the horizontal line cutting the ham in half, also as shown. For θ and $L(\theta)$ thus defined, let $B(\theta) =$ the amount of bread to the left of $L(\theta)$ minus that to the right of $L(\theta)$.

To understand this argument, one must understand what we mean by "to the left" or "to the right". Here, we mean to the left or right of the line as viewed in the direction θ. Imagine you are walking along the line in direction θ (directly right if $\theta = 0$, directly left if $\theta = \pi$, etc).

We will further accept the fact that B is continuous as a function of θ, which seems intuitively obvious. We need to prove that $B(c) = 0$ for some angle c.

Since $L(0)$ and $L(\pi)$ are drawn in opposite direction, $B(0) = -B(\pi)$. If $B(0) > 0$, we apply the IVT on $[0, \pi]$ with $B(0) > 0$, $B(\pi) < 0$, and B continuous on $[0, \pi]$; by IVT, $B(c) = 0$ for some $c \in [0, \pi]$. On the other hand, if $B(0) < 0$, then we apply the IVT with $B(0) < 0$ and $B(\pi) > 0$. If $B(0) = 0$, we are also done; $L(0)$ is the appropriate line.

2.9 The Formal Definition of a Limit

Preliminary Questions

1. Given that $\lim_{x\to 0} \cos x = 1$, which of the following statements is true?

(a) If $|\cos x - 1|$ is very small, then x is close to 0.

(b) There is an $\epsilon > 0$ such that if $0 < |\cos x - 1| < \epsilon$, then $|x| < 10^{-5}$.

(c) There is a $\delta > 0$ such that if $0 < |x| < \delta$, then $|\cos x - 1| < 10^{-5}$.

(d) There is a $\delta > 0$ such that if $0 < |x - 1| < \delta$, then $|\cos x| < 10^{-5}$.

SOLUTION The true statement is **(c)**: There is a $\delta > 0$ such that if $0 < |x| < \delta$, then $|\cos x - 1| < 10^{-5}$.

2. Suppose it is known that for a given ϵ and δ, if $0 < |x - 3| < \delta$, then $|f(x) - 2| < \epsilon$. Which of the following statements must also be true?

(a) If $0 < |x - 3| < 2\delta$, then $|f(x) - 2| < \epsilon$.

(b) If $0 < |x - 3| < \delta$, then$|f(x) - 2| < 2\epsilon$.

(c) If $0 < |x - 3| < \frac{\delta}{2}$, then $|f(x) - 2| < \frac{\epsilon}{2}$.

(d) If $0 < |x - 3| < \frac{\delta}{2}$, then $|f(x) - 2| < \epsilon$.

SOLUTION Statements **(b)** and **(d)** are true.

Exercises

1. Based on the information conveyed in Figure 5(A), find values of L, ϵ, and $\delta > 0$ such that the following statement holds: If $|x| < \delta$, then $|f(x) - L| < \epsilon$.

SOLUTION We see $-0.1 < x < 0.1$ forces $3.5 < f(x) < 4.8$. Rewritten, this means that $|x| < 0.1$ implies that $|f(x) - 4| < 0.8$. Looking at the limit definition $|x| < \delta$ implies $|f(x) - L| < \epsilon$, we can replace so that $L = 4$, $\epsilon = 0.8$, and $\delta = 0.1$.

2. Based on the information conveyed in Figure 5(B), find values of c, L, ϵ, and $\delta > 0$ such that the following statement holds: If $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.

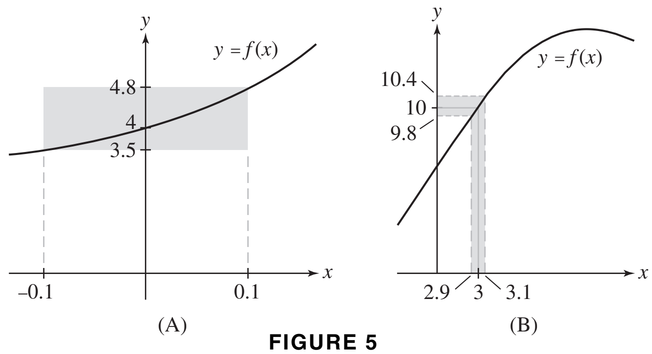
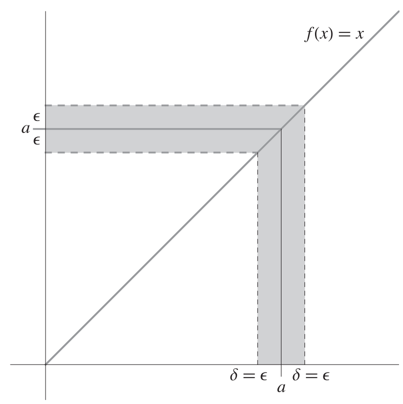


FIGURE 5

SOLUTION From the shaded region in the graph, we can see that when $2.9 < x < 3.1$, then $9.8 < f(x) < 10.4$. Rewriting these double inequalities as absolute value inequalities, we get $|x - 3| < 0.1$ implies $|f(x) - 10| < 0.4$. Replacing numbers where appropriate in the definition of the limit $|x - c| < \delta$ implies $|f(x) - L| < \epsilon$, we get $L = 10$, $\epsilon = 0.4$, $c = 3$, and $\delta = 0.1$.

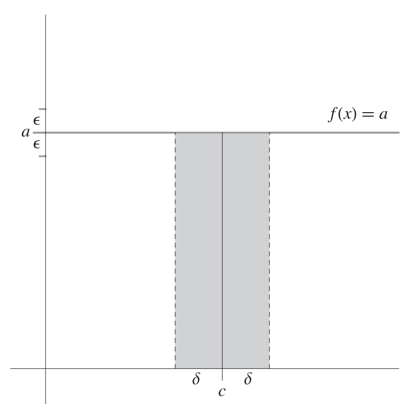
3. Make a sketch illustrating the following statement: To prove $\lim_{x \to a} x = a$, given $\epsilon > 0$, we can take $\delta = \epsilon$ to have the gap be small enough.

SOLUTION See the figure below. With $\delta = \epsilon$, the gap is within ϵ of a.



4. Make a sketch illustrating the following statement: To prove $\lim_{x \to c} a = a$, given $\epsilon > 0$, we can choose any $\delta > 0$ to have the gap be small enough.

SOLUTION See the figure below. With any choice for δ, the gap is within ϵ of a.



5. Consider $\lim_{x \to 4} f(x)$, where $f(x) = 8x + 3$.

(a) Show that $|f(x) - 35| = 8|x - 4|$.

(b) Show that for any $\epsilon > 0$, if $0 < |x - 4| < \delta$, then $|f(x) - 35| < \epsilon$, where $\delta = \frac{\epsilon}{8}$. Explain how this proves rigorously that $\lim_{x \to 4} f(x) = 35$.

SOLUTION

(a) $|f(x) - 35| = |8x + 3 - 35| = |8x - 32| = |8(x - 4)| = 8\,|x - 4|$. (Remember that the last step is justified because $8 > 0$.)

(b) Let $\epsilon > 0$. Let $\delta = \epsilon/8$ and suppose $|x - 4| < \delta$. By part **(a)**, $|f(x) - 35| = 8|x - 4| < 8\delta$. Substituting $\delta = \epsilon/8$, we see $|f(x) - 35| < 8\epsilon/8 = \epsilon$. We see that, for any $\epsilon > 0$, we found an appropriate δ so that $|x - 4| < \delta$ implies $|f(x) - 35| < \epsilon$. Hence $\lim_{x \to 4} f(x) = 35$.

6. Consider $\lim_{x\to 2} f(x)$, where $f(x) = 4x - 1$.

(a) Show that if $0 < |x-2| < \delta$, then $|f(x) - 7| < 4\delta$.

(b) Find a δ such that

$$\text{If} \quad 0 < |x-2| < \delta, \qquad \text{then} \quad |f(x) - 7| < 0.01$$

(c) Prove rigorously that $\lim_{x\to 2} f(x) = 7$.

SOLUTION

(a) If $0 < |x-2| < \delta$, then $|f(x) - 7| = |(4x-1) - 7| = 4|x-2| < 4\delta$.

(b) If $0 < |x-2| < \delta = .0025$, then $|(4x-1) - 7| = 4|x-2| < 4\delta = .01$.

(c) Let $\epsilon > 0$ be given. Then whenever $0 < |x-2| < \delta = \epsilon/4$, we have $|(4x-1) - 7| = 4|x-2| < 4\delta = \epsilon$. Since ϵ was arbitrary, we conclude that $\lim_{x\to 2}(4x-1) = 7$.

7. Consider $\lim_{x\to 2} x^2 = 4$ (refer to Example 2).

(a) Show that if $0 < |x-2| < 0.01$, then $|x^2 - 4| < 0.05$.

(b) Show that if $0 < |x-2| < 0.0002$, then $|x^2 - 4| < 0.0009$.

(c) Find a value of δ such that if $0 < |x-2| < \delta$, then $|x^2 - 4|$ is less than 10^{-4}.

SOLUTION

(a) If $0 < |x-2| < \delta = .01$, then $|x| < 3$ and $\left|x^2 - 4\right| = |x-2||x+2| \le |x-2|\,(|x|+2) < 5|x-2| < .05$.

(b) If $0 < |x-2| < \delta = .0002$, then $|x| < 2.0002$ and

$$\left|x^2 - 4\right| = |x-2||x+2| \le |x-2|\,(|x|+2) < 4.0002|x-2| < .00080004 < .0009$$

(c) Note that $\left|x^2 - 4\right| = |(x+2)(x-2)| \le |x+2|\,|x-2|$. Since $|x-2|$ can get arbitrarily small, we can require $|x-2| < 1$ so that $1 < x < 3$. This ensures that $|x+2|$ is at most 5. Now we know that $\left|x^2 - 4\right| \le 5|x-2|$. Let $\delta = 10^{-5}$. Then, if $|x-2| < \delta$, we get $\left|x^2 - 4\right| \le 5|x-2| < 5 \times 10^{-5} < 10^{-4}$ as desired.

8. Consider the limit $\lim_{x\to 5} x^2 = 25$.

(a) Show that if $4 < x < 6$, then $|x^2 - 25| < 11|x-5|$. *Hint:* Write $|x^2 - 25| = |x+5| \cdot |x-5|$.

(b) Find a δ such that if $0 < |x-5| < \delta$, then $|x^2 - 25| < 10^{-3}$.

(c) Give a rigorous proof of the limit by showing that if $0 < |x-5| < \delta$, then $|x^2 - 25| < \epsilon$, where δ is the smaller of $\frac{\epsilon}{11}$ and 1.

SOLUTION

(a) If $4 < x < 6$, then $|x-5| < \delta = 1$ and $\left|x^2 - 25\right| = |x-5||x+5| \le |x-5|\,(|x|+5) < 11|x-5|$.

(b) If $0 < |x-5| < \delta = \frac{.001}{11}$, then $x < 6$ and $\left|x^2 - 25\right| = |x-5||x+5| \le |x-5|\,(|x|+5) < 11|x-5| < .001$.

(c) Let $0 < |x-5| < \delta = \min\left\{1, \frac{\epsilon}{11}\right\}$. Since $\delta < 1$, $|x-5| < \delta < 1$ implies $4 < x < 6$. Specifically, $x < 6$ and

$$\left|x^2 - 25\right| = |x-5||x+5| \le |x-5|\,(|x|+5) < |x-5|(6+5) = 11|x-5|$$

Since δ is also less than $\epsilon/11$, we can conclude $11|x-5| < 11(\epsilon/11) = \epsilon$, thus completing the rigorous proof that if $|x-5| < \delta$, then $|x^2 - 25| < \epsilon$.

9. Refer to Example 3 to find a value of $\delta > 0$ such that

$$\text{If} \;\; 0 < |x-3| < \delta, \qquad \text{then} \;\; \left|\frac{1}{x} - \frac{1}{3}\right| < 10^{-4}$$

SOLUTION Example 3 shows that for any $\epsilon > 0$ we have

$$\left|\frac{1}{x} - \frac{1}{3}\right| \le \epsilon \quad \text{if } |x-3| < \delta$$

where δ is the smaller of the numbers 6ϵ and 1. In our case, we may take $\delta = 6 \times 10^{-4}$.

10. Use Figure 6 to find a value of $\delta > 0$ such that the following statement holds: If $0 < |x-2| < \delta$, then $\left|1/x^2 - \frac{1}{4}\right| < \epsilon$ for $\epsilon = 0.03$. Then find a value of δ that works for $\epsilon = 0.01$.

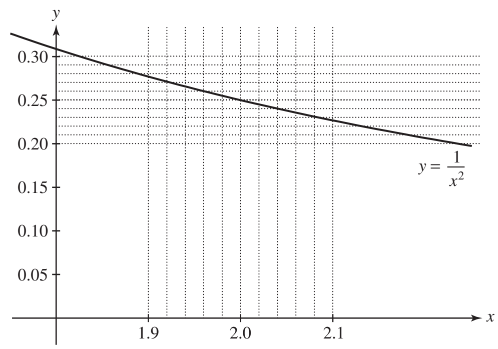
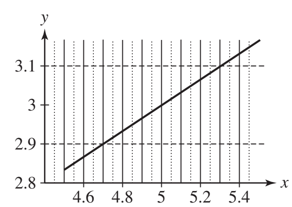


FIGURE 6

SOLUTION From Figure 6, it appears that $0 < |x - 2| < \delta = 0.1$ will guarantee that $\left|\frac{1}{x^2} - \frac{1}{4}\right| < \epsilon = 0.03$. It also appears that $0 < |x - 2| < \delta = 0.04$ will guarantee that $\left|\frac{1}{x^2} - \frac{1}{4}\right| < \epsilon = 0.01$.

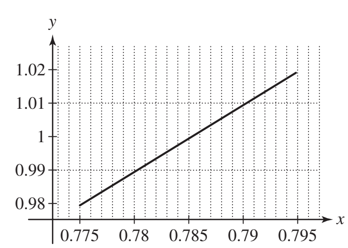
11. GU Plot $f(x) = \sqrt{2x - 1}$ together with the horizontal lines $y = 2.9$ and $y = 3.1$. Use this plot to find a value of $\delta > 0$ such that if $0 < |x - 5| < \delta$, then $|\sqrt{2x - 1} - 3| < 0.1$.

SOLUTION From the plot below, we see that $\delta = 0.25$ will guarantee that $|\sqrt{2x - 1} - 3| < 0.1$ whenever $|x - 5| \le \delta$.



12. GU Plot $f(x) = \tan x$ together with the horizontal lines $y = 0.99$ and $y = 1.01$. Use this plot to find a value of $\delta > 0$ such that if $0 < \left|x - \frac{\pi}{4}\right| < \delta$, then $|\tan x - 1| < 0.01$.

SOLUTION From the plot below, we see that $\delta = 0.005$ will guarantee that $|\tan x - 1| < 0.01$ whenever $|x - \frac{\pi}{4}| \le \delta$.

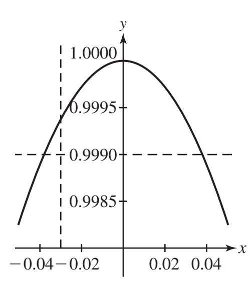


13. GU The function $f(x) = 2^{-x^2}$ satisfies $\lim_{x \to 0} f(x) = 1$. Use a plot of f to find a value of $\delta > 0$ such that $|f(x) - 1| < 0.001$ if $0 < |x| < \delta$.

SOLUTION From the plot below, we see that $\delta = 0.03$ will guarantee that

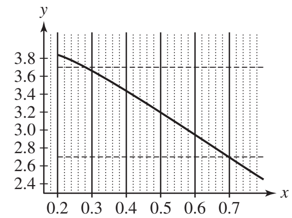
$$\left|2^{-x^2} - 1\right| < 0.001$$

whenever $0 < |x| < \delta$.

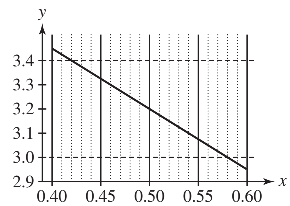


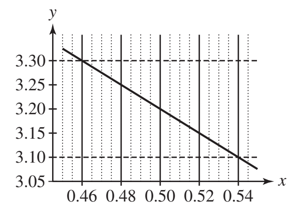
14. GU Let $f(x) = \frac{4}{x^2 + 1}$ and $\epsilon = 0.5$. Using a plot of f, find a value of $\delta > 0$ such that if $0 < \left|x - \frac{1}{2}\right| < \delta$, then $\left|f(x) - \frac{16}{5}\right| < \epsilon$. Repeat for $\epsilon = 0.2$ and 0.1.

SOLUTION From the plot below, we see that $\delta = 0.18$ will guarantee that $|f(x) - \frac{16}{5}| < 0.5$ whenever $|x - \frac{1}{2}| < \delta$.



When $\epsilon = 0.2$, we see that $\delta = 0.075$ will guarantee $|f(x) - \frac{16}{5}| < \epsilon$ whenever $|x - \frac{1}{2}| < \delta$ (examine the plot below at the left); when $\epsilon = 0.1$, $\delta = 0.035$ will guarantee $|f(x) - \frac{16}{5}| < \epsilon$ whenever $|x - \frac{1}{2}| < \delta$ (examine the plot below at the right).





15. Consider $\lim_{x\to 2} \frac{1}{x}$.

(a) Show that if $|x - 2| < 1$, then

$$\left|\frac{1}{x} - \frac{1}{2}\right| < \frac{1}{2}|x - 2|$$

(b) Find a $\delta > 0$ such that if $0 < |x - 2| < \delta$, then $\left|\frac{1}{x} - \frac{1}{2}\right| < 0.01$.

(c) Let δ be the smaller of 1 and 2ϵ. Prove the following:

$$\text{If } 0 < |x - 2| < \delta, \qquad \text{then } \left|\frac{1}{x} - \frac{1}{2}\right| < \epsilon$$

Then explain why this proves that $\lim_{x\to 2} \frac{1}{x} = \frac{1}{2}$.

SOLUTION

(a) Since $|x - 2| < 1$, it follows that $1 < x < 3$, in particular that $x > 1$. Because $x > 1$, then $\frac{1}{x} < 1$ and

$$\left|\frac{1}{x} - \frac{1}{2}\right| = \left|\frac{2 - x}{2x}\right| = \frac{|x - 2|}{2x} < \frac{1}{2}|x - 2|$$

(b) Choose $\delta = 0.02$. Then $\left|\frac{1}{x} - \frac{1}{2}\right| < \frac{1}{2}\delta = 0.01$ by part (a).

(c) Let $\delta = \min\{1, 2\epsilon\}$ and suppose that $|x - 2| < \delta$. Then by part (a) we have

$$\left|\frac{1}{x} - \frac{1}{2}\right| < \frac{1}{2}|x - 2| < \frac{1}{2}\delta < \frac{1}{2} \cdot 2\epsilon = \epsilon$$

Let $\epsilon > 0$ be given. Then whenever $0 < |x - 2| < \delta = \min\{1, 2\epsilon\}$, we have

$$\left|\frac{1}{x} - \frac{1}{2}\right| < \frac{1}{2}\delta \le \epsilon$$

Since ϵ was arbitrary, we conclude that $\lim_{x\to 2} \frac{1}{x} = \frac{1}{2}$.

16. Consider $\lim_{x\to 1} \sqrt{x + 3}$.

(a) Show that if $|x - 1| < 4$, then $|\sqrt{x + 3} - 2| < \frac{1}{2}|x - 1|$. *Hint:* Multiply the inequality by $|\sqrt{x + 3} + 2|$ and observe that $|\sqrt{x + 3} + 2| > 2$.

(b) Find $\delta > 0$ such that if $0 < |x - 1| < \delta$, then $|\sqrt{x + 3} - 2| < 10^{-4}$.

(c) Prove rigorously that the limit is equal to 2.

SOLUTION

(a) $|x-1|<4$ implies that $-3<x<5$. Since $x>-3$, then $\sqrt{x+3}$ is defined (and positive), whence

$$\left|\sqrt{x+3}-2\right| = \left|\frac{\left(\sqrt{x+3}-2\right)}{1}\frac{\left(\sqrt{x+3}+2\right)}{\left(\sqrt{x+3}+2\right)}\right| = \frac{|x-1|}{\sqrt{x+3}+2} < \frac{|x-1|}{2}$$

(b) Choose $\delta = .0002$. Then provided $0<|x-1|<\delta$, we have $x>-3$ and therefore

$$\left|\sqrt{x+3}-2\right| < \frac{|x-1|}{2} < \frac{\delta}{2} = .0001$$

by part (a).

(c) Let $\epsilon>0$ be given. Then whenever $0<|x-1|<\delta=\min\{2\epsilon,4\}$, we have $x>-3$ and thus

$$\left|\sqrt{x+3}-2\right| = \left|\frac{\left(\sqrt{x+3}-2\right)}{1}\frac{\left(\sqrt{x+3}+2\right)}{\left(\sqrt{x+3}+2\right)}\right| = \frac{|x-1|}{\sqrt{x+3}+2} < \frac{2\epsilon}{2} = \epsilon$$

Since ϵ was arbitrary, we conclude that $\lim_{x\to1}\sqrt{x+3}=2$.

17. Let $f(x)=\sin x$. Using a calculator, we find

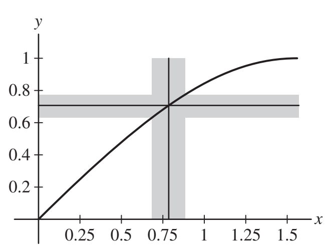
$$f\left(\frac{\pi}{4}-0.1\right)\approx 0.633,\quad f\left(\frac{\pi}{4}\right)\approx 0.707,\quad f\left(\frac{\pi}{4}+0.1\right)\approx 0.774$$

Use these values and the fact that f is increasing on $\left[0,\frac{\pi}{2}\right]$ to justify the statement

$$\text{If } 0<\left|x-\frac{\pi}{4}\right|<0.1, \qquad \text{then } \left|f(x)-f\left(\frac{\pi}{4}\right)\right|<0.08$$

Then draw a figure like Figure 3 to illustrate this statement.

SOLUTION Since $f(x)$ is increasing on the interval, the three $f(x)$ values tell us that $.633 \le f(x) \le .774$ for all x between $\frac{\pi}{4}-.1$ and $\frac{\pi}{4}+.1$. We may subtract $f(\frac{\pi}{4})$ from the inequality for $f(x)$. This shows that, for $\frac{\pi}{4}-.1<x<\frac{\pi}{4}+.1$, $.633-f(\frac{\pi}{4}) \le f(x)-f(\frac{\pi}{4}) \le .774-f(\frac{\pi}{4})$. This means that, if $|x-\frac{\pi}{4}|<.1$, then $.633-.707 \le f(x)-f(\frac{\pi}{4}) \le .774-.707$, so $-0.074 \le f(x)-f(\frac{\pi}{4}) \le 0.067$. Then $-0.08 < f(x)-f(\frac{\pi}{4}) < 0.08$ follows from this, so $|x-\frac{\pi}{4}|<0.1$ implies $|f(x)-f(\frac{\pi}{4})|<.08$. The figure below illustrates this.



18. Adapt the argument in Example 1 to prove rigorously that $\lim_{x\to c}(ax+b)=ac+b$, where a, b, c are arbitrary.

SOLUTION $|f(x)-(ac+b)| = |(ax+b)-(ac+b)| = |a(x-c)| = |a|\,|x-c|$. This says the gap is $|a|$ times as large as $|x-c|$. Let $\epsilon>0$. Let $\delta=\epsilon/(1+|a|)$, where we have added 1 to the denominator to avoid division by zero in the case $a=0$. If $|x-c|<\delta$, we get

$$|f(x)-(ac+b)| = |a|\,|x-c| < |a|\frac{\epsilon}{1+|a|} < \epsilon$$

which is what we had to prove.

19. Adapt the argument in Example 2 to prove rigorously that $\lim_{x\to c}x^2=c^2$ for all c.

SOLUTION To relate the gap to $|x-c|$, we take

$$\left|x^2-c^2\right| = |(x+c)(x-c)| = |x+c|\,|x-c|$$

We choose δ in two steps. First, since we are requiring $|x-c|$ to be small, we require $\delta < 1+|c|$, where we have added 1 to avoid complications associated with $c=0$. Then $|x| < 2|c|+1$ and $|x+c| < 3|c|+1$, so $|x-c||x+c| < (3|c|+1)\delta$. Next, we require that $\delta < \dfrac{\epsilon}{3|c|+1}$, so

$$|x-c||x+c| < \frac{\epsilon}{3|c|+1}(3|c|+1) = \epsilon$$

and we are done.

Therefore, given $\epsilon > 0$, we let

$$\delta = \min\left\{1+|c|, \frac{\epsilon}{3|c|+1}\right\}$$

Then, for $|x-c| < \delta$, we have

$$|x^2-c^2| = |x-c|\,|x+c| < (3|c|+1)\delta < (3|c|+1)\frac{\epsilon}{3|c|+1} = \epsilon.$$

20. Adapt the argument in Example 3 to prove rigorously that $\lim_{x\to c} x^{-1} = \frac{1}{c}$ for all $c \neq 0$.

SOLUTION To relate the gap to $|x-c|$, we find:

$$\left|x^{-1}-\frac{1}{c}\right| = \left|\frac{c-x}{cx}\right| = \frac{|x-c|}{|cx|}$$

Since $|x-c|$ is required to be small, we may assume from the outset that $|x-c| < |c|/2$, so that x is between $|c|/2$ and $3|c|/2$. This forces $|cx| > |c|/2$, from which

$$\frac{|x-c|}{|cx|} < \frac{2}{|c|}|x-c|$$

If $\delta < \epsilon(\frac{|c|}{2})$,

$$\left|x^{-1}-\frac{1}{c}\right| < \frac{2}{|c|}|x-c| < \frac{2}{|c|}\frac{|c|}{2}\epsilon = \epsilon$$

Therefore, given $\epsilon > 0$ we let

$$\delta = \min\left(\frac{|c|}{2}, \epsilon\left(\frac{|c|}{2}\right)\right)$$

We have shown that $|x^{-1}-\frac{1}{c}| < \epsilon$ if $0 < |x-c| < \delta$.

In Exercises 21–26, use the formal definition of the limit to prove the statement rigorously.

21. $\lim_{x\to 4} \sqrt{x} = 2$

SOLUTION Let $\epsilon > 0$ be given. We bound $|\sqrt{x}-2|$ by multiplying $\dfrac{\sqrt{x}+2}{\sqrt{x}+2}$.

$$|\sqrt{x}-2| = \left|(\sqrt{x}-2)\cdot\frac{\sqrt{x}+2}{\sqrt{x}+2}\right| = \left|\frac{x-4}{\sqrt{x}+2}\right| = |x-4|\left|\frac{1}{\sqrt{x}+2}\right|$$

We can assume $\delta < 1$, so that $|x-4| < 1$, and hence $\sqrt{x}+2 > \sqrt{3}+2 > 3$. This gives us

$$|\sqrt{x}-2| = |x-4|\left|\frac{1}{\sqrt{x}+2}\right| < |x-4|\frac{1}{3}.$$

Let $\delta = \min(1, 3\epsilon)$. If $|x-4| < \delta$,

$$|\sqrt{x}-2| = |x-4|\left|\frac{1}{\sqrt{x}+2}\right| < |x-4|\frac{1}{3} < \delta\frac{1}{3} < 3\epsilon\frac{1}{3} = \epsilon$$

thus proving the limit rigorously.

22. $\lim_{x\to 1}(3x^2+x) = 4$

SOLUTION Let $\epsilon > 0$ be given. We bound $\left|(3x^2 + x) - 4\right|$ using quadratic factoring.

$$\left|(3x^2 + x) - 4\right| = \left|3x^2 + x - 4\right| = |(3x + 4)(x - 1)| = |x - 1||3x + 4|$$

Let $\delta = \min(1, \frac{\epsilon}{10})$. Since $\delta < 1$, we get $|3x + 4| < 10$, so that

$$\left|(3x^2 + x) - 4\right| = |x - 1||3x + 4| < 10|x - 1|$$

Since $\delta < \frac{\epsilon}{10}$, we get

$$\left|(3x^2 + x) - 4\right| < 10|x - 1| < 10\frac{\epsilon}{10} = \epsilon$$

23. $\lim_{x\to 1} x^3 = 1$

SOLUTION Let $\epsilon > 0$ be given. We bound $\left|x^3 - 1\right|$ by factoring the difference of cubes:

$$\left|x^3 - 1\right| = \left|(x^2 + x + 1)(x - 1)\right| = |x - 1|\left|x^2 + x + 1\right|$$

Let $\delta = \min(1, \frac{\epsilon}{7})$, and assume $|x - 1| < \delta$. Since $\delta < 1$, $0 < x < 2$. Since $x^2 + x + 1$ increases as x increases for $x > 0$, $x^2 + x + 1 < 7$ for $0 < x < 2$, and so

$$\left|x^3 - 1\right| = |x - 1|\left|x^2 + x + 1\right| < 7|x - 1| < 7\frac{\epsilon}{7} = \epsilon$$

and the limit is rigorously proven.

24. $\lim_{x\to 0}(x^2 + x^3) = 0$

SOLUTION Let $\epsilon > 0$ be given. First, we bound $|x^2 + x^3 - 0| = |x^2 + x^3|$:

$$|x^2 + x^3| = |x| \cdot |x||1 + x|$$

Let $\delta = \min(1, \frac{\epsilon}{2})$, and suppose $|x - 0| = |x| < \delta$. Since $\delta < 1$, $-1 < x < 1$. This means $0 < 1 + x < 2$. Thus,

$$|x^2 + x^3| = |x| \cdot |x||1 + x| < 2|x| < 2 \cdot \frac{\epsilon}{2} = \epsilon$$

and the limit is rigorously proven.

25. $\lim_{x\to 2} x^{-2} = \dfrac{1}{4}$

SOLUTION Let $\epsilon > 0$ be given. First, we bound $\left|x^{-2} - \frac{1}{4}\right|$:

$$\left|x^{-2} - \frac{1}{4}\right| = \left|\frac{4 - x^2}{4x^2}\right| = |2 - x|\left|\frac{2 + x}{4x^2}\right|$$

Let $\delta = \min(1, \frac{4}{5}\epsilon)$, and suppose $|x - 2| < \delta$. Since $\delta < 1$, $|x - 2| < 1$, so $1 < x < 3$. This means that $4x^2 > 4$ and $|2 + x| < 5$, so that $\dfrac{2 + x}{4x^2} < \frac{5}{4}$. We get

$$\left|x^{-2} - \frac{1}{4}\right| = |2 - x|\left|\frac{2 + x}{4x^2}\right| < \frac{5}{4}|x - 2| < \frac{5}{4} \cdot \frac{4}{5}\epsilon = \epsilon$$

and the limit is rigorously proven.

26. $\lim_{x\to 0} x \sin\dfrac{1}{x} = 0$

SOLUTION Let $\epsilon > 0$ be given. Let $\delta = \epsilon$, and assume $|x - 0| = |x| < \delta$. We bound $x \sin\frac{1}{x}$.

$$\left|x \sin\frac{1}{x} - 0\right| = |x|\left|\sin\frac{1}{x}\right| < |x| < \delta = \epsilon$$

27. Let $f(x) = \dfrac{x}{|x|}$. Prove rigorously that $\lim_{x\to 0} f(x)$ does not exist. *Hint:* Show that for any L, there always exists some x such that $|x| < \delta$ but $|f(x) - L| \ge \frac{1}{2}$, no matter how small δ is taken.

SOLUTION Let L be any real number. Let $\delta > 0$ be any small positive number. Let $x = \frac{\delta}{2}$, which satisfies $|x| < \delta$, and $f(x) = 1$. We consider two cases:

- $(|f(x) - L| \ge \frac{1}{2})$: we are done.
- $(|f(x) - L| < \frac{1}{2})$: This means $\frac{1}{2} < L < \frac{3}{2}$. In this case, let $x = -\frac{\delta}{2}$. $f(x) = -1$, and so $\frac{3}{2} < L - f(x)$.

In either case, there exists an x such that $|x| < \frac{\delta}{2}$, but $|f(x) - L| \ge \frac{1}{2}$.

28. Prove rigorously that $\lim_{x\to 0} |x| = 0$.

SOLUTION Let $\epsilon > 0$ be given, and take $\delta = \epsilon$. Then, whenever $0 < |x - 0| = |x| < \delta$, it follows that

$$\Big||x| - 0\Big| = |x| < \delta = \epsilon$$

29. Let $f(x) = \min(x, x^2)$, where $\min(a, b)$ is the minimum of a and b. Prove rigorously that $\lim_{x\to 1} f(x) = 1$.

SOLUTION Let $\epsilon > 0$ be given, and take $\delta = \min(1, \frac{\epsilon}{2})$. Suppose $0 < |x - 1| < \delta$. Then either $1 - \delta < x < 1$ or $1 < x < 1 + \delta$. If $1 < x < 1 + \delta$, then

$$|f(x) - 1| = |x - 1| < \delta < \frac{\epsilon}{2} < \epsilon$$

as required. On the other hand, if $1 - \delta < x < 1$, then $|1 + x| < 2$ and

$$|f(x) - 1| = |x^2 - 1| = |x - 1||x + 1| < 2|x - 1| < 2\delta < \epsilon$$

Thus, $\lim_{x\to 1} f(x) = 1$.

30. Prove rigorously that $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist.

SOLUTION Let $\delta > 0$ be a given small positive number, and let L be any real number. We will prove that $\left|\sin\frac{1}{x} - L\right| \geq \frac{1}{2}$ for some x such that $|x| < \delta$.

Let $N > 0$ be a positive integer large enough so that $\frac{2}{(4N+1)\pi} < \delta$. Let

$$x_1 = \frac{2}{(4N + 1)\pi}$$

$$x_2 = \frac{2}{(4N + 3)\pi}$$

$$x_2 < x_1 < \delta$$

$$\sin\frac{1}{x_1} = \sin\frac{(4N + 1)\pi}{2} = 1 \quad \text{and} \quad \sin\frac{1}{x_2} = \sin\frac{(4N + 3)\pi}{2} = -1$$

If $|\sin\frac{1}{x_1} - L| \geq \frac{1}{2}$, we are done. Therefore, let's assume that $|\sin\frac{1}{x_1} - L| < \frac{1}{2}$. $-\frac{1}{2} < \sin\frac{1}{x_1} - L < \frac{1}{2}$, so $L - \frac{1}{2} < \sin\frac{1}{x_1} = 1 < L + \frac{1}{2}$. This means $L > \frac{1}{2}$, so that $|\sin\frac{1}{x_2} - L| = |-1 - L| > \frac{3}{2}$. In either case, there is an x such that $|x| < \delta$ but $|\sin\frac{1}{x} - L| \geq \frac{1}{2}$, so no limit L can exist.

31. Use the identity

$$\sin x + \sin y = 2\sin\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right)$$

to prove that

$$\sin(a + h) - \sin a = h\frac{\sin(h/2)}{h/2}\cos\left(a + \frac{h}{2}\right) \qquad \boxed{1}$$

Then use the inequality $\left|\frac{\sin x}{x}\right| \leq 1$ for $x \neq 0$ to show that

$|\sin(a + h) - \sin a| < |h|$ for all a. Finally, prove rigorously that $\lim_{x\to a} \sin x = \sin a$.

SOLUTION We first write

$$\sin(a + h) - \sin a = \sin(a + h) + \sin(-a)$$

Applying the identity with $x = a + h$, $y = -a$, yields:

$$\sin(a + h) - \sin a = \sin(a + h) + \sin(-a) = 2\sin\left(\frac{a + h - a}{2}\right)\cos\left(\frac{2a + h}{2}\right)$$

$$= 2\sin\left(\frac{h}{2}\right)\cos\left(a + \frac{h}{2}\right) = 2\left(\frac{h}{h}\right)\sin\left(\frac{h}{2}\right)\cos\left(a + \frac{h}{2}\right) = h\frac{\sin(h/2)}{h/2}\cos\left(a + \frac{h}{2}\right)$$

Therefore,

$$|\sin(a + h) - \sin a| = |h|\left|\frac{\sin(h/2)}{h/2}\right|\left|\cos\left(a + \frac{h}{2}\right)\right|$$

Using the fact that $\left|\frac{\sin\theta}{\theta}\right| < 1$ and that $|\cos\theta| \leq 1$, and making the substitution $h = x - a$, we see that this last relation is equivalent to

$$|\sin x - \sin a| < |x - a|$$

Now, to prove the desired limit, let $\epsilon > 0$, and take $\delta = \epsilon$. If $|x - a| < \delta$, then

$$|\sin x - \sin a| < |x - a| < \delta = \epsilon$$

Therefore, a δ was found for arbitrary ϵ, and the proof is complete.

Further Insights and Challenges

32. Uniqueness of the Limit Prove that a function converges to at most one limiting value. In other words, use the limit definition to prove that if $\lim_{x\to c} f(x) = L_1$ and $\lim_{x\to c} f(x) = L_2$, then $L_1 = L_2$.

SOLUTION Let $\epsilon > 0$ be given. Since $\lim_{x\to c} f(x) = L_1$, there exists δ_1 such that if $|x - c| < \delta_1$ then $|f(x) - L_1| < \epsilon$. Similarly, since $\lim_{x\to c} f(x) = L_2$, there exists δ_2 such that if $|x - c| < \delta_2$ then $|f(x) - L_2| < \epsilon$. Now let $|x - c| < \min(\delta_1, \delta_2)$ and observe that

$$\begin{aligned} |L_1 - L_2| &= |L_1 - f(x) + f(x) - L_2| \\ &\le |L_1 - f(x)| + |f(x) - L_2| \\ &= |f(x) - L_1| + |f(x) - L_2| < 2\epsilon \end{aligned}$$

So, $|L_1 - L_2| < 2\epsilon$ for any $\epsilon > 0$. We have $|L_1 - L_2| = \lim_{\epsilon\to 0} |L_1 - L_2| < \lim_{\epsilon\to 0} 2\epsilon = 0$. Therefore, $|L_1 - L_2| = 0$ and, hence, $L_1 = L_2$.

In Exercises 33–35, prove the statement using the formal limit definition.

33. The Constant Multiple Law [Theorem 1, part (ii) in Section 2.3]

SOLUTION Suppose that $\lim_{x\to c} f(x) = L$. We wish to prove that $\lim_{x\to c} af(x) = aL$.

Let $\epsilon > 0$ be given. $\epsilon/|a|$ is also a positive number. Since $\lim_{x\to c} f(x) = L$, we know there is a $\delta > 0$ such that $|x - c| < \delta$ forces $|f(x) - L| < \epsilon/|a|$. Suppose $|x - c| < \delta$. $|af(x) - aL| = |a||f(x) - aL| < |a|(\epsilon/|a|) = \epsilon$, so the rule is proven.

34. The Squeeze Theorem (Theorem 1 in Section 2.6)

SOLUTION *Proof of the Squeeze Theorem.* Suppose that (i) the inequalities $h(x) \le f(x) \le g(x)$ hold for all x near (but not equal to) a and (ii) $\lim_{x\to a} h(x) = \lim_{x\to a} g(x) = L$. Let $\epsilon > 0$ be given.

- By (i), there exists a $\delta_1 > 0$ such that $h(x) \le f(x) \le g(x)$ whenever $0 < |x - a| < \delta_1$.
- By (ii), there exist $\delta_2 > 0$ and $\delta_3 > 0$ such that $|h(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta_2$ and $|g(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta_3$.
- Choose $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then whenever $0 < |x - a| < \delta$ we have $L - \epsilon < h(x) \le f(x) \le g(x) < L + \epsilon$; that is, $|f(x) - L| < \epsilon$. Since ϵ was arbitrary, we conclude that $\lim_{x\to a} f(x) = L$.

35. The Product Law [Theorem 1, part (iii) in Section 2.3]. *Hint:* Use the identity.

$$f(x)g(x) - LM = (f(x) - L)\,g(x) + L(g(x) - M)$$

SOLUTION Before we can prove the Product Law, we need to establish one preliminary result. We are given that $\lim_{x\to c} g(x) = M$. Consequently, if we set $\epsilon = 1$, then the definition of a limit guarantees the existence of a $\delta_1 > 0$ such that whenever $0 < |x - c| < \delta_1$, $|g(x) - M| < 1$. Applying the inequality $|g(x)| - |M| \le |g(x) - M|$, it follows that $|g(x)| < 1 + |M|$. In other words, because $\lim_{x\to c} g(x) = M$, there exists a $\delta_1 > 0$ such that $|g(x)| < 1 + |M|$ whenever $0 < |x - c| < \delta_1$.

We can now prove the Product Law. Let $\epsilon > 0$. As proven above, because $\lim_{x\to c} g(x) = M$, there exists a $\delta_1 > 0$ such that $|g(x)| < 1 + |M|$ whenever $0 < |x - c| < \delta_1$. Furthermore, by the definition of a limit, $\lim_{x\to c} g(x) = M$ implies there exists a $\delta_2 > 0$ such that $|g(x) - M| < \frac{\epsilon}{2(1+|L|)}$ whenever $0 < |x - c| < \delta_2$. We have included the "1+" in the denominator to avoid division by zero in case $L = 0$. The reason for including the factor of 2 in the denominator will become clear shortly. Finally, because $\lim_{x\to c} f(x) = L$, there exists a $\delta_3 > 0$ such that $|f(x) - L| < \frac{\epsilon}{2(1+|M|)}$ whenever $0 < |x - c| < \delta_3$. Now, let $\delta = \min(\delta_1, \delta_2, \delta_3)$. Then, for all x satisfying $0 < |x - c| < \delta$, we have

$$\begin{aligned} |f(x)g(x) - LM| &= |(f(x) - L)g(x) + L(g(x) - M)| \\ &\le |f(x) - L|\,|g(x)| + |L|\,|g(x) - M| \\ &< \frac{\epsilon}{2(1 + |M|)}(1 + |M|) + |L|\frac{\epsilon}{2(1 + |L|)} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Hence,

$$\lim_{x\to c} f(x)g(x) = LM = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$$

36. Let $f(x) = 1$ if x is rational and $f(x) = 0$ if x is irrational. Prove that $\lim_{x\to c} f(x)$ does not exist for any c. *Hint:* There exist rational and irrational numbers arbitrarily close to any c.

SOLUTION Let c be any number, and let $\delta > 0$ be an arbitrary small number. We will prove that there is an x such that $|x - c| < \delta$, but $|f(x) - f(c)| > \frac{1}{2}$. c must be either irrational or rational. If c is rational, then $f(c) = 1$. Since the irrational numbers are dense, there is at least one irrational number z such that $|z - c| < \delta$. $|f(z) - f(c)| = 1 > \frac{1}{2}$, so the function is discontinuous at $x = c$. On the other hand, if c is irrational, then there is a *rational* number q such that $|q - c| < \delta$. $|f(q) - f(c)| = |1 - 0| = 1 > \frac{1}{2}$, so the function is discontinuous at $x = c$.

37. Here is a function with strange continuity properties:

$$f(x) = \begin{cases} \dfrac{1}{q} & \text{if } x \text{ is the rational number } p/q \text{ in lowest terms} \\ 0 & \text{if } x \text{ is an irrational number} \end{cases}$$

(a) Show that f is discontinuous at c if c is rational. *Hint:* There exist irrational numbers arbitrarily close to c.

(b) Show that f is continuous at c if c is irrational. *Hint:* Let I be the interval $\{x : |x - c| < 1\}$. Show that for any $Q > 0$, I contains at most finitely many fractions p/q with $q < Q$. Conclude that there is a δ such that all fractions in $\{x : |x - c| < \delta\}$ have a denominator larger than Q.

SOLUTION

(a) Let c be any rational number and suppose that, in lowest terms, $c = p/q$, where p and q are integers. To prove the discontinuity of f at c, we must show there is an $\epsilon > 0$ such that for any $\delta > 0$ there is an x for which $|x - c| < \delta$, but that $|f(x) - f(c)| > \epsilon$. Let $\epsilon = \frac{1}{2q}$ and $\delta > 0$. Since there is at least one irrational number between any two distinct real numbers, there is some irrational x between c and $c + \delta$. Hence, $|x - c| < \delta$, but $|f(x) - f(c)| = |0 - \frac{1}{q}| = \frac{1}{q} > \frac{1}{2q} = \epsilon$.

(b) Let c be irrational, let $\epsilon > 0$ be given, and let $N > 0$ be a prime integer sufficiently large so that $\frac{1}{N} < \epsilon$. Let $\frac{p_1}{q_1}, \dots, \frac{p_m}{q_m}$ be all rational numbers $\frac{p}{q}$ in lowest terms such that $|\frac{p}{q} - c| < 1$ and $q < N$. Since N is finite, this is a finite list; hence, one number $\frac{p_i}{q_i}$ in the list must be closest to c. Let $\delta = \frac{1}{2}|\frac{p_i}{q_i} - c|$. By construction, $|\frac{p_i}{q_i} - c| > \delta$ for all $i = 1 \dots m$. Therefore, for any rational number $\frac{p}{q}$ such that $|\frac{p}{q} - c| < \delta$, $q > N$, so $\frac{1}{q} < \frac{1}{N} < \epsilon$.

Therefore, for any *rational* number x such that $|x - c| < \delta$, $|f(x) - f(c)| < \epsilon$. $|f(x) - f(c)| = 0$ for any irrational number x, so $|x - c| < \delta$ implies that $|f(x) - f(c)| < \epsilon$ for any number x.

38. Write a formal definition of the following:

$$\lim_{x\to\infty} f(x) = L$$

SOLUTION $\lim_{x\to\infty} f(x) = L$ if, for any $\epsilon > 0$, there exists an $M > 0$ such that $|f(x) - L| < \epsilon$ whenever $x > M$.

39. Write a formal definition of the following:

$$\lim_{x\to a} f(x) = \infty$$

SOLUTION $\lim_{x\to a} f(x) = \infty$ if, for any $M > 0$, there exists a $\delta > 0$ such that $f(x) > M$ whenever $0 < |x - a| < \delta$.

CHAPTER REVIEW EXERCISES

1. The position of a particle at time t (s) is $s(t) = \sqrt{t^2 + 1}$ m. Compute its average velocity over $[2, 5]$ and estimate its instantaneous velocity at $t = 2$.

SOLUTION Let $s(t) = \sqrt{t^2 + 1}$. The average velocity over $[2, 5]$ is

$$\frac{s(5) - s(2)}{5 - 2} = \frac{\sqrt{26} - \sqrt{5}}{3} \approx 0.954 \text{ m/s}$$

From the data in the table below, we estimate that the instantaneous velocity at $t = 2$ is approximately 0.894 m/s.

interval	[1.9, 2]	[1.99, 2]	[1.999, 2]	[2, 2.001]	[2, 2.01]	[2, 2.1]
average ROC	0.889769	0.893978	0.894382	0.894472	0.894873	0.898727

2. A rock dropped from a state of rest at time $t = 0$ on the planet Ginormon travels a distance $s(t) = 15.2t^2$ m in t seconds. Estimate the instantaneous velocity at $t = 5$.

SOLUTION To estimate the instantaneous velocity at $t = 5$, we examine the following table.

time interval	[4.99, 5]	[4.999, 5]	[4.9999, 5]	[5, 5.0001]	[5, 5.001]	[5, 5.01]
average velocity	151.848	151.9848	151.99848	152.00152	152.0152	152.152

The instantaneous velocity at $t = 5$ is approximately 152.0 m/s.

3. For $f(x) = \sqrt{2x}$ compute the slopes of the secant lines from 16 to each of $16 \pm 0.01, 16 \pm 0.001, 16 \pm 0.0001$ and use those values to estimate the slope of the tangent line at $x = 16$.

SOLUTION

x interval	[15.99, 16]	[15.999, 16]	[15.9999, 16]	[16, 16.0001]	[16, 16.001]	[16, 16.01]
slope of secant	0.176804	0.176779	0.176777	0.176776	0.176774	0.176749

The slope of the tangent line at $t = 16$ is approximately 0.1768.

4. Show that the slope of the secant line for $f(x) = x^3 - 2x$ over $[5, x]$ is equal to $x^2 + 5x + 23$. Use this to estimate the slope of the tangent line at $x = 5$.

SOLUTION Let $f(x) = x^3 - 2x$. The slope of the secant line over the interval $[5, x]$ is

$$\frac{f(x) - f(5)}{x - 5} = \frac{x^3 - 2x - 115}{x - 5} = \frac{(x - 5)(x^2 + 5x + 23)}{x - 5} = x^2 + 5x + 23$$

provided $x \neq 5$. To estimate the slope of the tangent line at $x = 5$, examine the values in the table below.

x	4.99	4.999	4.9999	5.0001	5.001	5.01
slope of secant	72.8501	72.985001	72.998500	73.001500	73.015001	73.1501

The slope of the tangent line at $x = 5$ is approximately 73.0.

In Exercises 5–10, estimate the limit numerically to two decimal places or state that the limit does not exist.

5. $\displaystyle\lim_{x\to 0} \frac{1 - \cos^3(x)}{x^2}$

SOLUTION Let $f(x) = \frac{1-\cos^3 x}{x^2}$. The data in the table below suggests that

$$\lim_{x\to 0} \frac{1 - \cos^3 x}{x^2} \approx 1.50$$

In constructing the table, we take advantage of the fact that f is an even function.

x	±0.001	±0.01	±0.1
$f(x)$	1.500000	1.499912	1.491275

(The exact value is $\frac{3}{2}$.)

6. $\displaystyle\lim_{x\to 1} x^{1/(x-1)}$

SOLUTION Let $f(x) = x^{1/(x-1)}$. The data in the table below suggests that

$$\lim_{x\to 1} x^{1/(x-1)} \approx 2.72$$

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	2.867972	2.731999	2.719642	2.716924	2.704814	2.593742

(The exact value is e.)

7. $\displaystyle\lim_{x\to 2} \frac{x^x - 4}{x^2 - 4}$

SOLUTION Let $f(x) = \frac{x^x-4}{x^2-4}$. The data in the table below suggests that

$$\lim_{x\to 2} \frac{x^x - 4}{x^2 - 4} \approx 1.69$$

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	1.575461	1.680633	1.691888	1.694408	1.705836	1.828386

(The exact value is $1 + \ln 2$.)

8. $\lim_{x\to 2} \frac{x-2}{2^x-4}$

SOLUTION Let $f(x) = \frac{x-2}{2^x-4}$. The data in the table below suggests that

$$\lim_{x\to 2} \frac{x-2}{2^x-4} \approx 0.36.$$

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	0.37332	0.36193	0.36080	0.36055	0.35943	0.34832

(The exact value is $\frac{1}{4\ln 2}$.)

9. $\lim_{x\to 1} \left(\frac{7}{1-x^7} - \frac{3}{1-x^3}\right)$

SOLUTION Let $f(x) = \frac{7}{1-x^7} - \frac{3}{1-x^3}$. The data in the table below suggests that

$$\lim_{x\to 1} \left(\frac{7}{1-x^7} - \frac{3}{1-x^3}\right) \approx 2.00$$

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	2.347483	2.033498	2.003335	1.996668	1.966835	1.685059

(The exact value is 2.)

10. $\lim_{x\to 2} \frac{3^x-9}{5^x-25}$

SOLUTION Let $f(x) = \frac{3^x-9}{5^x-25}$. The data in the table below suggests that

$$\lim_{x\to 2} \frac{3^x-9}{5^x-25} \approx 0.246$$

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	0.251950	0.246365	0.245801	0.245675	0.245110	0.239403

(The exact value is $\frac{9}{25}\frac{\ln 3}{\ln 5}$.)

In Exercises 11–50, evaluate the limit if it exists. If not, determine whether the one-sided limits exist. For limits that don't exist indicate whether they can be expressed as "$= -\infty$" or "$= \infty$".

11. $\lim_{x\to 4}(3 + x^{1/2})$

SOLUTION $\lim_{x\to 4}(3 + x^{1/2}) = 3 + \sqrt{4} = 5$

12. $\lim_{x\to 1} \frac{5-x^2}{4x+7}$

SOLUTION $\lim_{x\to 1} \frac{5-x^2}{4x+7} = \frac{5-1^2}{4(1)+7} = \frac{4}{11}$

13. $\lim_{x\to -2} \frac{4}{x^3}$

SOLUTION $\lim_{x\to -2} \frac{4}{x^3} = \frac{4}{(-2)^3} = -\frac{1}{2}$

14. $\lim_{x\to -1} \frac{3x^2+4x+1}{x+1}$

SOLUTION $\lim_{x\to -1} \frac{3x^2+4x+1}{x+1} = \lim_{x\to -1} \frac{(3x+1)(x+1)}{x+1} = \lim_{x\to -1}(3x+1) = 3(-1)+1 = -2$

15. $\lim_{t\to 9} \frac{\sqrt{t}-3}{t-9}$

SOLUTION $\lim_{t\to 9}\frac{\sqrt{t}-3}{t-9}=\lim_{t\to 9}\frac{\sqrt{t}-3}{(\sqrt{t}-3)(\sqrt{t}+3)}=\lim_{t\to 9}\frac{1}{\sqrt{t}+3}=\frac{1}{\sqrt{9}+3}=\frac{1}{6}$

16. $\lim_{x\to 3}\frac{\sqrt{x+1}-2}{x-3}$

SOLUTION

$$\lim_{x\to 3}\frac{\sqrt{x+1}-2}{x-3}=\lim_{x\to 3}\frac{\sqrt{x+1}-2}{x-3}\cdot\frac{\sqrt{x+1}+2}{\sqrt{x+1}+2}=\lim_{x\to 3}\frac{(x+1)-4}{(x-3)(\sqrt{x+1}+2)}$$
$$=\lim_{x\to 3}\frac{1}{\sqrt{x+1}+2}=\frac{1}{\sqrt{3+1}+2}=\frac{1}{4}$$

17. $\lim_{x\to 1}\frac{x^3-x}{x-1}$

SOLUTION $\lim_{x\to 1}\frac{x^3-x}{x-1}=\lim_{x\to 1}\frac{x(x-1)(x+1)}{x-1}=\lim_{x\to 1}x(x+1)=1(1+1)=2$

18. $\lim_{h\to 0}\frac{2(a+h)^2-2a^2}{h}$

SOLUTION

$$\lim_{h\to 0}\frac{2(a+h)^2-2a^2}{h}=\lim_{h\to 0}\frac{2a^2+4ah+2h^2-2a^2}{h}=\lim_{h\to 0}\frac{h(4a+2h)}{h}=\lim_{h\to 0}(4a+2h)=4a+2(0)=4a$$

19. $\lim_{t\to 9}\frac{t-6}{\sqrt{t}-3}$

SOLUTION As $t\to 9$, the numerator $t-6\to 3\neq 0$ while the denominator $\sqrt{t}-3\to 0$. Accordingly,

$$\lim_{t\to 9}\frac{t-6}{\sqrt{t}-3}\qquad\text{does not exist.}$$

Similarly, the one-sided limits as $t\to 9^-$ and as $t\to 9^+$ also do not exist. Let's take a closer look at the limit as $t\to 9^-$. The numerator approaches a positive number while the denominator $\sqrt{t}-3\to 0^-$. We may therefore express this one-sided limit as

$$\lim_{t\to 9-}\frac{t-6}{\sqrt{t}-3}=-\infty$$

On the other hand, as $t\to 9^+$, the numerator approaches a positive number while the denominator $\sqrt{t}-3\to 0^+$, so we can express this one-sided limit as

$$\lim_{t\to 9+}\frac{t-6}{\sqrt{t}-3}=\infty$$

Because one of the one-sided limits approaches $-\infty$ and the other approaches ∞, the two-sided limit can be expressed neither as "$=-\infty$" nor as "$=\infty$".

20. $\lim_{s\to 0}\frac{1-\sqrt{s^2+1}}{s^2}$

SOLUTION

$$\lim_{s\to 0}\frac{1-\sqrt{s^2+1}}{s^2}=\lim_{s\to 0}\frac{1-\sqrt{s^2+1}}{s^2}\cdot\frac{1+\sqrt{s^2+1}}{1+\sqrt{s^2+1}}=\lim_{s\to 0}\frac{1-(s^2+1)}{s^2(1+\sqrt{s^2+1})}$$
$$=\lim_{s\to 0}\frac{-1}{1+\sqrt{s^2+1}}=\frac{-1}{1+\sqrt{0^2+1}}=-\frac{1}{2}$$

21. $\lim_{x\to -1^+}\frac{1}{x+1}$

SOLUTION As $x\to -1^+$, the numerator remains constant at $1\neq 0$ while the denominator $x+1\to 0$. Accordingly,

$$\lim_{x\to -1^+}\frac{1}{x+1}\qquad\text{does not exist.}$$

Taking a closer look at the denominator, we see that $x+1\to 0^+$ as $x\to -1^+$. Because the numerator is also approaching a positive number, we may express this limit as

$$\lim_{x\to -1^+}\frac{1}{x+1}=\infty$$

22. $\lim_{y\to\frac{1}{3}} \frac{3y^2+5y-2}{6y^2-5y+1}$

SOLUTION $\lim_{y\to\frac{1}{3}} \frac{3y^2+5y-2}{6y^2-5y+1} = \lim_{y\to\frac{1}{3}} \frac{(3y-1)(y+2)}{(3y-1)(2y-1)} = \lim_{y\to\frac{1}{3}} \frac{y+2}{2y-1} = -7$

23. $\lim_{x\to 1} \frac{x^3-2x}{x-1}$

SOLUTION As $x \to 1$, the numerator $x^3 - 2x \to -1 \neq 0$ while the denominator $x - 1 \to 0$. Accordingly,

$$\lim_{x\to 1} \frac{x^3-2x}{x-1} \qquad \text{does not exist.}$$

Similarly, the one-sided limits as $x \to 1^-$ and as $x \to 1^+$ also do not exist. Let's take a closer look at the limit as $x \to 1^-$. The numerator approaches a negative number while the denominator $x - 1 \to 0^-$. We may therefore express this one-sided limit as

$$\lim_{x\to 1-} \frac{x^3-2x}{x-1} = \infty$$

On the other hand, as $x \to 1^+$, the numerator approaches a negative number while the denominator $x - 1 \to 0^+$, so we can express this one-sided limit as

$$\lim_{x\to 1+} \frac{x^3-2x}{x-1} = -\infty$$

Because one of the one-sided limits approaches $-\infty$ and the other approaches ∞, the two-sided limit can be expressed neither as "$= -\infty$" nor as "$= \infty$".

24. $\lim_{a\to b} \frac{a^2-3ab+2b^2}{a-b}$

SOLUTION $\lim_{a\to b} \frac{a^2-3ab+2b^2}{a-b} = \lim_{a\to b} \frac{(a-b)(a-2b)}{a-b} = \lim_{a\to b}(a-2b) = b - 2b = -b$

25. $\lim_{x\to 0} \frac{4^{3x}-4^x}{4^x-1}$

SOLUTION

$$\lim_{x\to 0} \frac{4^{3x}-4^x}{4^x-1} = \lim_{x\to 0} \frac{4^x(4^x-1)(4^x+1)}{4^x-1} = \lim_{x\to 0} 4^x(4^x+1) = 1 \cdot 2 = 2.$$

26. $\lim_{\theta\to 0} \frac{\sin 5\theta}{\theta}$

SOLUTION

$$\lim_{\theta\to 0} \frac{\sin 5\theta}{\theta} = 5 \lim_{\theta\to 0} \frac{\sin 5\theta}{5\theta} = 5(1) = 5$$

27. $\lim_{x\to 1.5} \left\lfloor \frac{1}{x} \right\rfloor$

SOLUTION $\lim_{x\to 1.5} \left\lfloor \frac{1}{x} \right\rfloor = \left\lfloor \frac{1}{1.5} \right\rfloor = \left\lfloor \frac{2}{3} \right\rfloor = 0$

28. $\lim_{\theta\to\frac{\pi}{4}} \sec\theta$

SOLUTION $\lim_{\theta\to\frac{\pi}{4}} \sec\theta = \sec\frac{\pi}{4} = \sqrt{2}$

29. $\lim_{z\to -3} \frac{z+3}{z^2+4z+3}$

SOLUTION $\lim_{z\to -3} \frac{z+3}{z^2+4z+3} = \lim_{z\to -3} \frac{z+3}{(z+3)(z+1)} = \lim_{z\to -3} \frac{1}{z+1} = -\frac{1}{2}$

30. $\lim_{x\to 1} \frac{x^3-ax^2+ax-1}{x-1}$

SOLUTION Using

$$x^3 - ax^2 + ax - 1 = (x-1)(x^2+x+1) - ax(x-1) = (x-1)(x^2+x-ax+1)$$

we find

$$\lim_{x\to 1}\frac{x^3-ax^2+ax-1}{x-1} = \lim_{x\to 1}\frac{(x-1)(x^2+x-ax+1)}{x-1} = \lim_{x\to 1}(x^2+x-ax+1)$$
$$= 1^2+1-a(1)+1 = 3-a$$

31. $\lim_{x\to b}\frac{x^3-b^3}{x-b}$

SOLUTION $\lim_{x\to b}\frac{x^3-b^3}{x-b} = \lim_{x\to b}\frac{(x-b)(x^2+xb+b^2)}{x-b} = \lim_{x\to b}(x^2+xb+b^2) = b^2+b(b)+b^2 = 3b^2$

32. $\lim_{x\to 0}\frac{\sin 4x}{\sin 3x}$

SOLUTION

$$\lim_{x\to 0}\frac{\sin 4x}{\sin 3x} = \frac{4}{3}\lim_{x\to 0}\frac{\sin 4x}{4x}\cdot\frac{3x}{\sin 3x} = \frac{4}{3}\lim_{x\to 0}\frac{\sin 4x}{4x}\cdot\lim_{x\to 0}\frac{3x}{\sin 3x} = \frac{4}{3}(1)(1) = \frac{4}{3}$$

33. $\lim_{x\to 0}\left(\frac{1}{3x}-\frac{1}{x(x+3)}\right)$

SOLUTION $\lim_{x\to 0}\left(\frac{1}{3x}-\frac{1}{x(x+3)}\right) = \lim_{x\to 0}\frac{(x+3)-3}{3x(x+3)} = \lim_{x\to 0}\frac{1}{3(x+3)} = \frac{1}{3(0+3)} = \frac{1}{9}$

34. $\lim_{\theta\to\frac{1}{4}} 3^{\tan(\pi\theta)}$

SOLUTION $\lim_{\theta\to\frac{1}{4}} 3^{\tan(\pi\theta)} = 3^{\tan(\pi/4)} = 3^1 = 3$

35. $\lim_{x\to 0^-}\frac{\lfloor x\rfloor}{x}$

SOLUTION For x sufficiently close to zero but negative, $\lfloor x\rfloor = -1$. Therefore, as $x\to 0^-$, the numerator $\lfloor x\rfloor \to -1 \neq 0$ while the denominator $x\to 0$. Accordingly,

$$\lim_{x\to 0^-}\frac{\lfloor x\rfloor}{x} \qquad \text{does not exist.}$$

Taking a closer look at the denominator, we see that $x\to 0^-$ as $x\to 0^-$. Because the numerator is also approaching a negative number, we may express this limit as

$$\lim_{x\to 0^-}\frac{\lfloor x\rfloor}{x} = \infty$$

36. $\lim_{x\to 0^+}\frac{\lfloor x\rfloor}{x}$

SOLUTION For x sufficiently close to zero but positive, $\lfloor x\rfloor = 0$. Therefore,

$$\lim_{x\to 0^+}\frac{\lfloor x\rfloor}{x} = \lim_{x\to 0^+}\frac{0}{x} = 0$$

37. $\lim_{\theta\to\frac{\pi}{2}}\theta\sec\theta$

SOLUTION First note that

$$\theta\sec\theta = \frac{\theta}{\cos\theta}$$

As $\theta\to\frac{\pi}{2}$, the numerator $\theta\to\frac{\pi}{2}\neq 0$ while the denominator $\cos\theta\to 0$. Accordingly,

$$\lim_{\theta\to\frac{\pi}{2}}\theta\sec\theta = \lim_{\theta\to\frac{\pi}{2}}\frac{\theta}{\cos\theta} \qquad \text{does not exist.}$$

Similarly, the one-sided limits as $\theta \to \frac{\pi}{2}^-$ and as $\theta \to \frac{\pi}{2}^+$ also do not exist. Let's take a closer look at the limit as $\theta \to \frac{\pi}{2}^-$. The numerator approaches a positive number while the denominator $\cos\theta \to 0^+$. We may therefore express this one-sided limit as

$$\lim_{\theta\to\frac{\pi}{2}^-} \theta\sec\theta = \lim_{\theta\to\frac{\pi}{2}^-} \frac{\theta}{\cos\theta} = \infty$$

On the other hand, as $\theta \to \frac{\pi}{2}^+$, the numerator approaches a positive number while the denominator $\cos\theta \to 0^-$, so we can express this one-sided limit as

$$\lim_{\theta\to\frac{\pi}{2}^+} \theta\sec\theta = \lim_{\theta\to\frac{\pi}{2}^+} \frac{\theta}{\cos\theta} = -\infty$$

Because one of the one-sided limits approaches $-\infty$ and the other approaches ∞, the two-sided limit can be expressed neither as "$= -\infty$" nor as "$= \infty$".

38. $\lim_{y\to3}\left(\sin\frac{\pi}{y}\right)^{-1/2}$

SOLUTION

$$\lim_{y\to3}\left(\sin\frac{\pi}{y}\right)^{-1/2} = \left(\sin\frac{\pi}{3}\right)^{-1/2} = \left(\frac{2}{\sqrt{3}}\right)^{1/2} = \frac{\sqrt{2}}{\sqrt[4]{3}}$$

39. $\lim_{\theta\to0}\frac{\cos\theta-2}{\theta}$

SOLUTION As $\theta \to 0$, the numerator $\cos\theta - 2 \to -1 \neq 0$ while the denominator $\theta \to 0$. Accordingly,

$$\lim_{\theta\to0}\frac{\cos\theta-2}{\theta} \qquad \text{does not exist.}$$

Similarly, the one-sided limits as $\theta \to 0^-$ and as $\theta \to 0^+$ also do not exist. Let's take a closer look at the limit as $\theta \to 0^-$. The numerator approaches a negative number while the denominator $\theta \to 0^-$. We may therefore express this one-sided limit as

$$\lim_{\theta\to0^-}\frac{\cos\theta-2}{\theta} = \infty$$

On the other hand, as $\theta \to 0^+$, the numerator approaches a negative number while the denominator $\theta \to 0^+$, so we can express this one-sided limit as

$$\lim_{\theta\to0^+}\frac{\cos\theta-2}{\theta} = -\infty$$

Because one of the one-sided limits approaches $-\infty$ and the other approaches ∞, the two-sided limit can be expressed neither as "$= -\infty$" nor as "$= \infty$".

40. $\lim_{x\to4.3}\frac{1}{x-\lfloor x\rfloor}$

SOLUTION $\lim_{x\to4.3}\frac{1}{x-\lfloor x\rfloor} = \frac{1}{4.3-\lfloor 4.3\rfloor} = \frac{1}{0.3} = \frac{10}{3}$

41. $\lim_{x\to2^-}\frac{x-3}{x-2}$

SOLUTION As $x \to 2^-$, the numerator $x - 3 \to -1 \neq 0$ while the denominator $x - 2 \to 0$. Accordingly,

$$\lim_{x\to2^+}\frac{x-3}{x-2} \qquad \text{does not exist.}$$

Taking a closer look at the denominator, we see that $x - 2 \to 0^-$ as $x \to 2^-$. Because the numerator is also approaching a negative number, we may express this limit as

$$\lim_{x\to2^-}\frac{x-3}{x-2} = \infty$$

42. $\lim_{t\to0}\frac{\sin^2 t}{t^3}$

SOLUTION First note that

$$\frac{\sin^2 t}{t^3} = \frac{\sin t}{t} \cdot \frac{\sin t}{t} \cdot \frac{1}{t}.$$

As $t \to 0$, each factor of $\frac{\sin t}{t}$ approaches 1; however, the numerator of the last factor remains constant at $1 \neq 0$ while the denominator of the last factor $t \to 0$. Accordingly,

$$\lim_{t \to 0} \frac{\sin^2 t}{t^3} \qquad \text{does not exist.}$$

Similarly, the one-sided limits as $t \to 0^-$ and as $t \to 0^+$ also do not exist. Let's take a closer look at the limit as $t \to 0^-$. The numerator of the last factor approaches a positive number while the denominator of the last factor $t \to 0^-$. We may therefore express this one-sided limit as

$$\lim_{t \to 0^-} \frac{\sin^2 t}{t^3} = -\infty$$

On the other hand, as $t \to 0^+$, the numerator of the last factor approaches a positive number while the denominator of the last factor $t \to 0^+$, so we can express this one-sided limit as

$$\lim_{t \to 0^+} \frac{\sin^2 t}{t^3} = \infty$$

Because one of the one-sided limits approaches $-\infty$ and the other approaches ∞, the two-sided limit can be expressed neither as "$= -\infty$" nor as "$= \infty$".

43. $\lim_{x \to 1^+} \left(\frac{1}{\sqrt{x-1}} - \frac{1}{\sqrt{x^2-1}} \right)$

SOLUTION First note that

$$\frac{1}{\sqrt{x-1}} - \frac{1}{\sqrt{x^2-1}} = \frac{1}{\sqrt{x-1}} \cdot \frac{\sqrt{x+1}}{\sqrt{x+1}} - \frac{1}{\sqrt{x^2-1}} = \frac{\sqrt{x+1}-1}{\sqrt{x^2-1}}$$

As $x \to 1^+$, the numerator $\sqrt{x+1} - 1 \to \sqrt{2} - 1 \neq 0$ while the denominator $\sqrt{x^2-1} \to 0$. Accordingly,

$$\lim_{x \to 1^+} \left(\frac{1}{\sqrt{x-1}} - \frac{1}{\sqrt{x^2-1}} \right) \qquad \text{does not exist.}$$

Taking a closer look at the denominator, we see that $\sqrt{x^2-1} \to 0^+$ as $x \to 1^+$. Because the numerator is also approaching a positive number, we may express this limit as

$$\lim_{x \to 1^+} \left(\frac{1}{\sqrt{x-1}} - \frac{1}{\sqrt{x^2-1}} \right) = \infty$$

44. $\lim_{t \to \frac{\pi}{2}} \sqrt{2t}(\cos t - 1)$

SOLUTION

$$\lim_{t \to \frac{\pi}{2}} \sqrt{2t}(\cos t - 1) = \lim_{t \to \frac{\pi}{2}} \sqrt{2t} \cdot \lim_{t \to \frac{\pi}{2}} (\cos t - 1) = \sqrt{\pi}\left(\cos \frac{\pi}{2} - 1\right) = -\sqrt{\pi}.$$

45. $\lim_{x \to \frac{\pi}{2}} \tan x$

SOLUTION First note that

$$\tan x = \frac{\sin x}{\cos x}$$

As $x \to \frac{\pi}{2}$, the numerator $\sin x \to 1 \neq 0$ while the denominator $\cos x \to 0$. Accordingly,

$$\lim_{x \to \frac{\pi}{2}} \tan x = \lim_{x \to \frac{\pi}{2}} \frac{\sin x}{\cos x} \qquad \text{does not exist.}$$

Similarly, the one-sided limits as $x \to \frac{\pi}{2}^-$ and as $x \to \frac{\pi}{2}^+$ also do not exist. Let's take a closer look at the limit as $x \to \frac{\pi}{2}^-$. The numerator approaches a positive number while the denominator $\cos x \to 0^+$. We may therefore express this one-sided limit as

$$\lim_{x \to \frac{\pi}{2}^-} \tan x = \lim_{x \to \frac{\pi}{2}^-} \frac{\sin x}{\cos x} = \infty$$

On the other hand, as $x \to \frac{\pi}{2}^+$, the numerator approaches a positive number while the denominator $\cos x \to 0^-$, so we can express this one-sided limit as

$$\lim_{x\to\frac{\pi}{2}^+} \tan x = \lim_{x\to\frac{\pi}{2}^+} \frac{\sin x}{\cos x} = -\infty$$

Because one of the one-sided limits approaches $-\infty$ and the other approaches ∞, the two-sided limit can be expressed neither as "$= -\infty$" nor as "$= \infty$".

46. $\lim_{t\to 0} \cos \frac{1}{t}$

SOLUTION As $t \to 0$, $\frac{1}{t}$ grows without bound and $\cos(\frac{1}{t})$ oscillates faster and faster. Consequently,

$$\lim_{t\to 0} \cos\left(\frac{1}{t}\right) \qquad \text{does not exist.}$$

Similarly, the one-sided limits as $t \to 0^-$ and as $t \to 0^+$ also do not exist. None of these limits can be expressed as "$= -\infty$" or as "$= \infty$".

47. $\lim_{t\to 0^+} \sqrt{t} \cos \frac{1}{t}$

SOLUTION For $t > 0$,

$$-1 \le \cos\left(\frac{1}{t}\right) \le 1$$

so

$$-\sqrt{t} \le \sqrt{t}\cos\left(\frac{1}{t}\right) \le \sqrt{t}$$

Because

$$\lim_{t\to 0^+} -\sqrt{t} = \lim_{t\to 0^+} \sqrt{t} = 0$$

it follows from the Squeeze Theorem that

$$\lim_{t\to 0^+} \sqrt{t}\cos\left(\frac{1}{t}\right) = 0$$

48. $\lim_{x\to 5^+} \frac{x^2-24}{x^2-25}$

SOLUTION As $x \to 5^+$, the numerator $x^2 - 24 \to 1 \ne 0$ while the denominator $x^2 - 25 \to 0$. Accordingly,

$$\lim_{x\to 5^+} \frac{x^2-24}{x^2-25} \qquad \text{does not exist.}$$

Taking a closer look at the denominator, we see that $x^2 - 25 \to 0^+$ as $x \to 5^+$. Because the numerator is also approaching a positive number, we may express this limit as

$$\lim_{x\to 5^+} \frac{x^2-24}{x^2-25} = \infty$$

49. $\lim_{x\to 0} \frac{\cos x - 1}{\sin x}$

SOLUTION

$$\lim_{x\to 0} \frac{\cos x - 1}{\sin x} = \lim_{x\to 0} \frac{\cos x - 1}{\sin x} \cdot \frac{\cos x + 1}{\cos x + 1} = \lim_{x\to 0} \frac{-\sin^2 x}{\sin x(\cos x + 1)} = -\lim_{x\to 0} \frac{\sin x}{\cos x + 1} = -\frac{0}{1+1} = 0$$

50. $\lim_{\theta\to 0} \frac{\tan\theta - \sin\theta}{\sin^3\theta}$

SOLUTION

$$\lim_{\theta\to 0} \frac{\tan\theta - \sin\theta}{\sin^3\theta} = \lim_{\theta\to 0} \frac{\sec\theta - 1}{\sin^2\theta} = \lim_{\theta\to 0} \frac{\sec\theta - 1}{\sin^2\theta} \cdot \frac{\sec\theta + 1}{\sec\theta + 1} = \lim_{\theta\to 0} \frac{\tan^2\theta}{\sin^2\theta(\sec\theta + 1)}$$

$$= \lim_{\theta\to 0} \frac{\sec^2\theta}{\sec\theta + 1} = \frac{1}{1+1} = \frac{1}{2}$$

51. Find the left- and right-hand limits of the function f in Figure 1 at $x = 0, 2, 4$. State whether f is left- or right-continuous (or both) at these points.

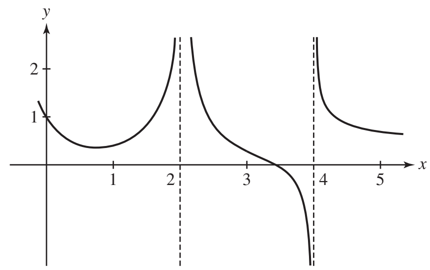


FIGURE 1

SOLUTION According to the graph of $f(x)$,

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = 1$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = \infty$$

$$\lim_{x\to 4^-} f(x) = -\infty$$

$$\lim_{x\to 4^+} f(x) = \infty$$

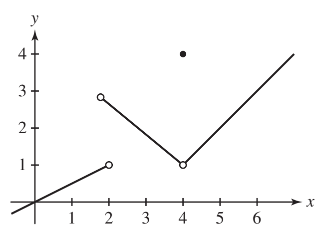
The function is both left- and right-continuous at $x = 0$ and neither left- nor right-continuous at $x = 2$ and $x = 4$.

52. Sketch the graph of a function f such that

(a) $\lim_{x\to 2^-} f(x) = 1, \qquad \lim_{x\to 2^+} f(x) = 3$

(b) $\lim_{x\to 4} f(x)$ exists but does not equal $f(4)$.

SOLUTION

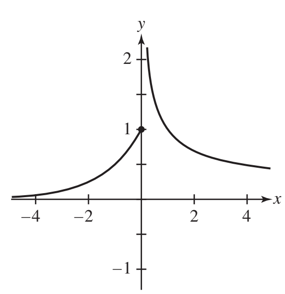


53. Graph h and describe the discontinuity:

$$h(x) = \begin{cases} 2^x & \text{for } x \le 0 \\ x^{-1/2} & \text{for } x > 0 \end{cases}$$

Is h left- or right-continuous?

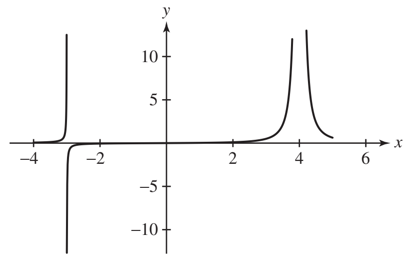
SOLUTION The graph of $h(x)$ is shown below. At $x = 0$, the function has an infinite discontinuity but is left-continuous.



54. Sketch the graph of a function g such that

$$\lim_{x\to -3^-} g(x) = \infty, \qquad \lim_{x\to -3^+} g(x) = -\infty, \qquad \lim_{x\to 4} g(x) = \infty$$

SOLUTION



55. Find the points of discontinuity of

$$g(x) = \begin{cases} \cos\left(\frac{\pi x}{2}\right) & \text{for } |x| < 1 \\ |x-1| & \text{for } |x| \geq 1 \end{cases}$$

Determine the type of discontinuity and whether g is left- or right-continuous.

SOLUTION First note that $\cos\left(\frac{\pi x}{2}\right)$ is continuous for $-1 < x < 1$ and that $|x-1|$ is continuous for $x \leq -1$ and for $x \geq 1$. Thus, the only points at which $g(x)$ might be discontinuous are $x = \pm 1$. At $x = 1$, we have

$$\lim_{x\to 1^-} g(x) = \lim_{x\to 1^-} \cos\left(\frac{\pi x}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$$

and

$$\lim_{x\to 1^+} g(x) = \lim_{x\to 1^+} |x-1| = |1-1| = 0$$

so $g(x)$ is continuous at $x = 1$. On the other hand, at $x = -1$,

$$\lim_{x\to -1^+} g(x) = \lim_{x\to -1^+} \cos\left(\frac{\pi x}{2}\right) = \cos\left(-\frac{\pi}{2}\right) = 0$$

and

$$\lim_{x\to -1^-} g(x) = \lim_{x\to -1^-} |x-1| = |-1-1| = 2$$

so $g(x)$ has a jump discontinuity at $x = -1$. Since $g(-1) = 2$, $g(x)$ is left-continuous at $x = -1$.

56. Find a constant b such that h is continuous at $x = 2$, where

$$h(x) = \begin{cases} x+1 & \text{for } |x| < 2 \\ b - x^2 & \text{for } |x| \geq 2 \end{cases}$$

With this choice of b, find all points of discontinuity.

SOLUTION To make $h(x)$ continuous at $x = 2$, we must have the two one-sided limits as x approaches 2 be equal. With

$$\lim_{x\to 2^-} h(x) = \lim_{x\to 2^-} (x+1) = 2 + 1 = 3$$

and

$$\lim_{x\to 2^+} h(x) = \lim_{x\to 2^+} (b - x^2) = b - 4$$

it follows that we must choose $b = 7$. Because $x + 1$ is continuous for $-2 < x < 2$ and $7 - x^2$ is continuous for $x \leq -2$ and for $x \geq 2$, the only possible point of discontinuity is $x = -2$. At $x = -2$,

$$\lim_{x\to -2^+} h(x) = \lim_{x\to -2^+} (x+1) = -2 + 1 = -1$$

and

$$\lim_{x\to -2^-} h(x) = \lim_{x\to -2^-} (7 - x^2) = 7 - (-2)^2 = 3$$

so $h(x)$ has a jump discontinuity at $x = -2$.

In Exercises 57–64, find the horizontal asymptotes of the function by computing the limits at infinity.

57. $f(x) = \dfrac{9x^2 - 4}{2x^2 - x}$

SOLUTION Because

$$\lim_{x\to\infty} \frac{9x^2-4}{2x^2-x} = \lim_{x\to\infty} \frac{9-4/x^2}{2-1/x} = \frac{9}{2}$$

and

$$\lim_{x\to-\infty} \frac{9x^2-4}{2x^2-x} = \lim_{x\to-\infty} \frac{9-4/x^2}{2-1/x} = \frac{9}{2}$$

it follows that the graph of $y = \dfrac{9x^2-4}{2x^2-x}$ has a horizontal asymptote of $y = \dfrac{9}{2}$.

58. $f(x) = \dfrac{x^2-3x^4}{x-1}$

SOLUTION Because

$$\lim_{x\to\infty} \frac{x^2-3x^4}{x-1} = \lim_{x\to\infty} \frac{x-3x^3}{1-1/x} = -\infty$$

and

$$\lim_{x\to-\infty} \frac{x^2-3x^4}{x-1} = \lim_{x\to-\infty} \frac{x-3x^3}{1-1/x} = \infty$$

it follows that the graph of $y = \dfrac{x^2-3x^4}{x-1}$ does not have any horizontal asymptotes.

59. $f(u) = \dfrac{8u-3}{\sqrt{16u^2+6}}$

SOLUTION Because

$$\lim_{u\to\infty} \frac{8u-3}{\sqrt{16u^2+6}} = \lim_{u\to\infty} \frac{8-3/u}{\sqrt{16+6/u^2}} = \frac{8}{\sqrt{16}} = 2$$

and

$$\lim_{u\to-\infty} \frac{8u-3}{\sqrt{16u^2+6}} = \lim_{u\to-\infty} \frac{8-3/u}{-\sqrt{16+6/u^2}} = \frac{8}{-\sqrt{16}} = -2$$

it follows that the graph of $y = \dfrac{8u-3}{\sqrt{16u^2+6}}$ has horizontal asymptotes of $y = \pm 2$.

60. $f(u) = \dfrac{2u^2-1}{\sqrt{6+u^4}}$

SOLUTION Because

$$\lim_{u\to\infty} \frac{2u^2-1}{\sqrt{6+u^4}} = \lim_{u\to\infty} \frac{2-1/u^2}{\sqrt{6/u^4+1}} = \frac{2}{\sqrt{1}} = 2$$

and

$$\lim_{u\to-\infty} \frac{2u^2-1}{\sqrt{6+u^4}} = \lim_{u\to-\infty} \frac{2-1/u^2}{\sqrt{6/u^4+1}} = \frac{2}{\sqrt{1}} = 2$$

it follows that the graph of $y = \dfrac{2u^2-1}{\sqrt{6+u^4}}$ has a horizontal asymptote of $y = 2$.

61. $f(x) = \dfrac{3x^{2/3}+9x^{3/7}}{7x^{4/5}-4x^{-1/3}}$

SOLUTION Because

$$\lim_{x\to\infty} \frac{3x^{2/3}+9x^{3/7}}{7x^{4/5}-4x^{-1/3}} = \lim_{x\to\infty} \frac{3x^{-2/15}+9x^{-13/35}}{7-4x^{-17/15}} = 0$$

and

$$\lim_{x\to-\infty} \frac{3x^{2/3}+9x^{3/7}}{7x^{4/5}-4x^{-1/3}} = \lim_{x\to-\infty} \frac{3x^{-2/15}+9x^{-13/35}}{7-4x^{-17/15}} = 0$$

it follows that the graph of $y = \dfrac{3x^{2/3}+9x^{3/7}}{7x^{4/5}-4x^{-1/3}}$ has a horizontal asymptote of $y = 0$.

62. $f(t) = \dfrac{t^{1/3} - t^{-1/3}}{(t - t^{-1})^{1/3}}$

SOLUTION Because

$$\lim_{t\ to \infty} \frac{t^{1/3} - t^{-1/3}}{(t - t^{-1})^{1/3}} = \lim_{t\to\infty} \frac{1 - t^{-2/3}}{(1 - t^{-2})^{1/3}} = \frac{1}{1^{1/3}} = 1$$

and

$$\lim_{t\ to -\infty} \frac{t^{1/3} - t^{-1/3}}{(t - t^{-1})^{1/3}} = \lim_{t\to -\infty} \frac{1 - t^{-2/3}}{(1 - t^{-2})^{1/3}} = \frac{1}{1^{1/3}} = 1$$

it follows that the graph of $y = \dfrac{t^{1/3} - t^{-1/3}}{(t - t^{-1})^{1/3}}$ has a horizontal asymptote of $y = 1$.

63. $f(t) = \frac{17}{1+2^t}$

SOLUTION Because $\lim_{t\to -\infty} 2^t = 0$ and $\lim_{t\to\infty} 2^t = \infty$, it follows that

$$\lim_{t\to -\infty} \frac{17}{1 + 2^t} = \frac{17}{1 + 0} = 17 \quad \text{and} \quad \lim_{t\to\infty} \frac{17}{1 + 2^t} = 0$$

The graph of $y = \dfrac{17}{1 + 2^t}$ has horizontal asymptotes of $y = 17$ and $y = 0$.

64. $g(x) = \frac{6}{1-3^{2x}}$

SOLUTION Because

$$\lim_{x\to -\infty} \frac{6}{1 - 3^{2x}} = \frac{6}{1 - 0} = 6 \quad \text{and} \quad \lim_{x\to\infty} \frac{6}{1 - 3^{2x}} = 0$$

it follows that the graph of $y = \frac{6}{1-3^{2x}}$ has horizontal asymptotes of $y = 6$ and $y = 0$.

65. Calculate (a)–(d), assuming that

$$\lim_{x\to 3} f(x) = 6, \qquad \lim_{x\to 3} g(x) = 4$$

(a) $\lim_{x\to 3}(f(x) - 2g(x))$

(b) $\lim_{x\to 3} x^2 f(x)$

(c) $\lim_{x\to 3} \dfrac{f(x)}{g(x) + x}$

(d) $\lim_{x\to 3}(2g(x)^3 - g(x)^{3/2})$

SOLUTION

(a) $\lim_{x\to 3}(f(x) - 2g(x)) = \lim_{x\to 3} f(x) - 2\lim_{x\to 3} g(x) = 6 - 2(4) = -2$

(b) $\lim_{x\to 3} x^2 f(x) = \lim_{x\to 3} x^2 \cdot \lim_{x\to 3} f(x) = 3^2 \cdot 6 = 54$

(c) $\lim_{x\to 3} \dfrac{f(x)}{g(x) + x} = \dfrac{\lim_{x\to 3} f(x)}{\lim_{x\to 3}(g(x) + x)} = \dfrac{6}{\lim_{x\to 3} g(x) + \lim_{x\to 3} x} = \dfrac{6}{4 + 3} = \dfrac{6}{7}$

(d) $\lim_{x\to 3}(2g(x)^3 - g(x)^{3/2}) = 2\left(\lim_{x\to 3} g(x)\right)^3 - \left(\lim_{x\to 3} g(x)\right)^{3/2} = 2(4)^3 - 4^{3/2} = 120$

66. Assume that the following limits exist:

$$A = \lim_{x\to a} f(x), \qquad B = \lim_{x\to a} g(x), \qquad L = \lim_{x\to a} \frac{f(x)}{g(x)}$$

Prove that if $L = 1$, then $A = B$. *Hint:* You cannot use the Quotient Law if $B = 0$, so apply the Product Law to L and B instead.

SOLUTION Suppose the limits A, B, and L all exist and $L = 1$. Then

$$B = B \cdot 1 = B \cdot L = \lim_{x\to a} g(x) \cdot \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} g(x) \frac{f(x)}{g(x)} = \lim_{x\to a} f(x) = A$$

67. In the notation of Exercise 66, give an example where L exists but neither A nor B exists.

SOLUTION Suppose

$$f(x) = \frac{1}{(x - a)^3} \qquad \text{and} \qquad g(x) = \frac{1}{(x - a)^5}$$

Then, neither A nor B exists, but

$$L = \lim_{x\to a} \frac{(x - a)^{-3}}{(x - a)^{-5}} = \lim_{x\to a}(x - a)^2 = 0$$

68. True or false?

(a) If $\lim_{x\to 3} f(x)$ exists, then $\lim_{x\to 3} f(x) = f(3)$.

(b) If $\lim_{x\to 0} \dfrac{f(x)}{x} = 1$, then $f(0) = 0$.

(c) If $\lim_{x\to -7} f(x) = 8$, then $\lim_{x\to -7} \dfrac{1}{f(x)} = \dfrac{1}{8}$.

(d) If $\lim_{x\to 5^+} f(x) = 4$ and $\lim_{x\to 5^-} f(x) = 8$, then $\lim_{x\to 5} f(x) = 6$.

(e) If $\lim_{x\to 0} \dfrac{f(x)}{x} = 1$, then $\lim_{x\to 0} f(x) = 0$.

(f) If $\lim_{x\to 5} f(x) = 2$, then $\lim_{x\to 5} f(x)^3 = 8$.

SOLUTION

(a) False. The limit $\lim_{x\to 3} f(x)$ may exist and need not equal $f(3)$. The limit is equal to $f(3)$ if $f(x)$ is continuous at $x = 3$.

(b) False. The value of the limit $\lim_{x\to 0} \dfrac{f(x)}{x} = 1$ does not depend on the value $f(0)$, so $f(0)$ can have any value.

(c) True, by the Limit Laws.

(d) False. If the two one-sided limits are not equal, then the two-sided limit does not exist.

(e) True. Apply the Product Law to the functions $\dfrac{f(x)}{x}$ and x.

(f) True, by the Limit Laws.

69. Let $f(x) = \left\lfloor \frac{1}{x} \right\rfloor$, where $\lfloor x \rfloor$ is the greatest integer function. Show that for $x \neq 0$,

$$\frac{1}{x} - 1 < \left\lfloor \frac{1}{x} \right\rfloor \leq \frac{1}{x}$$

Then use the Squeeze Theorem to prove that

$$\lim_{x\to 0} x \left\lfloor \frac{1}{x} \right\rfloor = 1$$

Hint: Treat the one-sided limits separately.

SOLUTION Let y be any real number. From the definition of the greatest integer function, it follows that $y - 1 < \lfloor y \rfloor \leq y$, with equality holding if and only if y is an integer. If $x \neq 0$, then $\frac{1}{x}$ is a real number, so

$$\frac{1}{x} - 1 < \left\lfloor \frac{1}{x} \right\rfloor \leq \frac{1}{x}$$

Upon multiplying this inequality through by x, we find

$$1 - x < x \left\lfloor \frac{1}{x} \right\rfloor \leq 1$$

Because

$$\lim_{x\to 0} (1 - x) = \lim_{x\to 0} 1 = 1$$

it follows from the Squeeze Theorem that

$$\lim_{x\to 0} x \left\lfloor \frac{1}{x} \right\rfloor = 1$$

70. Let r_1 and r_2 be the roots of $f(x) = ax^2 - 2x + 20$. Observe that f "approaches" the linear function $L(x) = -2x + 20$ as $a \to 0$. Because $r = 10$ is the unique root of L, we might expect one of the roots of f to approach 10 as $a \to 0$ (Figure 2). Prove that the roots can be labeled so that $\lim_{a\to 0} r_1 = 10$ and $\lim_{a\to 0} r_2 = \infty$.

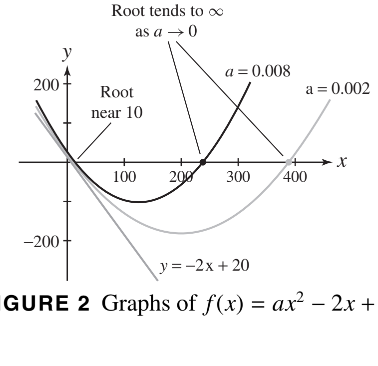


FIGURE 2 Graphs of $f(x) = ax^2 - 2x + 20$.

SOLUTION Using the quadratic formula, we find that the roots of the quadratic polynomial $f(x) = ax^2 - 2x + 20$ are

$$\frac{2 \pm \sqrt{4 - 80a}}{2a} = \frac{1 \pm \sqrt{1 - 20a}}{a} = \frac{20}{1 \pm \sqrt{1 - 20a}}$$

Now let

$$r_1 = \frac{20}{1 + \sqrt{1 - 20a}} \quad \text{and} \quad r_2 = \frac{20}{1 - \sqrt{1 - 20a}}$$

It is straightforward to calculate that

$$\lim_{a\to 0} r_1 = \lim_{a\to 0} \frac{20}{1 + \sqrt{1 - 20a}} = \frac{20}{2} = 10$$

and that

$$\lim_{a\to 0} r_2 = \lim_{a\to 0} \frac{20}{1 - \sqrt{1 - 20a}} = \infty$$

as desired.

71. Use the IVT to prove that the curves $y = x^2$ and $y = \cos x$ intersect.

SOLUTION Let $f(x) = x^2 - \cos x$. Note that any root of $f(x)$ corresponds to a point of intersection between the curves $y = x^2$ and $y = \cos x$. Now, $f(x)$ is continuous over the interval $[0, \frac{\pi}{2}]$, $f(0) = -1 < 0$ and $f(\frac{\pi}{2}) = \frac{\pi^2}{4} > 0$. Therefore, by the Intermediate Value Theorem, there exists a $c \in (0, \frac{\pi}{2})$ such that $f(c) = 0$; consequently, the curves $y = x^2$ and $y = \cos x$ intersect.

72. Use the IVT to prove that $f(x) = x^3 - \dfrac{x^2 + 2}{\cos x + 2}$ has a root in the interval $[0, 2]$.

SOLUTION Let $f(x) = x^3 - \frac{x^2+2}{\cos x+2}$. Because $\cos x + 2$ is never zero, $f(x)$ is continuous for all real numbers. Because

$$f(0) = -\frac{2}{3} < 0 \quad \text{and} \quad f(2) = 8 - \frac{6}{\cos 2 + 2} \approx 4.21 > 0$$

the Intermediate Value Theorem guarantees there exists a $c \in (0, 2)$ such that $f(c) = 0$.

73. Use the IVT to show that $2^{-x^2} = x$ has a solution on $(0, 1)$.

SOLUTION Let $f(x) = 2^{-x^2} - x$. Observe that f is continuous on $[0, 1]$ with $f(0) = 2^0 - 0 = 1 > 0$ and $f(1) = 2^{-1} - 1 < 0$. Therefore, the IVT guarantees there exists a $c \in (0, 1)$ such that $f(c) = 2^{-c^2} - c = 0$.

74. Use the Bisection Method to locate a solution of $x^2 - 7 = 0$ to two decimal places.

SOLUTION Let $f(x) = x^2 - 7$. By trial and error, we find that $f(2.6) = -0.24 < 0$ and $f(2.7) = 0.29 > 0$. Because $f(x)$ is continuous on $[2.6, 2.7]$, it follows from the Intermediate Value Theorem that $f(x)$ has a root on $(2.6, 2.7)$. We approximate the root by the midpoint of the interval: $x = 2.65$. Now, $f(2.65) = 0.0225 > 0$. Because $f(2.6)$ and $f(2.65)$ are of opposite sign, the root must lie on $(2.6, 2.65)$. The midpoint of this interval is $x = 2.625$ and $f(2.625) < 0$; hence, the root must be on the interval $(2.625, 2.65)$. Continuing in this fashion, we construct the following sequence of intervals and midpoints.

interval	midpoint
(2.625, 2.65)	2.6375
(2.6375, 2.65)	2.64375
(2.64375, 2.65)	2.646875
(2.64375, 2.646875)	2.6453125
(2.6453125, 2.646875)	2.64609375

At this point, we note that, to two decimal places, one root of $x^2 - 7 = 0$ is 2.65.

75. Give an example of a (discontinuous) function that does not satisfy the conclusion of the IVT on $[-1, 1]$. Then show that the function

$$f(x) = \begin{cases} \sin \dfrac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

satisfies the conclusion of the IVT on every interval $[-a, a]$.

SOLUTION Let $g(x) = \lfloor x \rfloor$. This function is discontinuous on $[-1, 1]$ with $g(-1) = -1$ and $g(1) = 1$. For all $c \in (-1, 1)$, $c \neq 0$, there is no x such that $g(x) = c$; thus, $g(x)$ does not satisfy the conclusion of the Intermediate Value Theorem on $[-1, 1]$.

Now, let

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

and let $a > 0$. On the interval

$$x \in \left[\frac{a}{2 + 2\pi a}, \frac{a}{2}\right] \subset [-a, a]$$

$\frac{1}{x}$ runs from $\frac{2}{a}$ to $\frac{2}{a} + 2\pi$, so the sine function covers one full period and clearly takes on every value from $-\sin a$ through $\sin a$.

76. Let $f(x) = \dfrac{1}{x + 2}$.

(a) Show that if $|x - 2| < 1$, then $\left|f(x) - \frac{1}{4}\right| < \dfrac{|x - 2|}{12}$. *Hint:* Observe that if $|x - 2| < 1$, then $|4(x + 2)| > 12$.

(b) Find $\delta > 0$ such that if $|x - 2| < \delta$, then $\left|f(x) - \frac{1}{4}\right| < 0.01$.

(c) Prove rigorously that $\lim_{x\to 2} f(x) = \frac{1}{4}$.

SOLUTION

(a) Let $f(x) = \frac{1}{x+2}$. Then

$$\left|f(x) - \frac{1}{4}\right| = \left|\frac{1}{x + 2} - \frac{1}{4}\right| = \left|\frac{4 - (x + 2)}{4(x + 2)}\right| = \frac{|x - 2|}{|4(x + 2)|}$$

If $|x - 2| < 1$, then $1 < x < 3$, so $3 < x + 2 < 5$ and $12 < 4(x + 2) < 20$. Hence,

$$\frac{1}{|4(x + 2)|} < \frac{1}{12} \qquad \text{and} \qquad \left|f(x) - \frac{1}{4}\right| < \frac{|x - 2|}{12}$$

(b) If $|x - 2| < \delta$, then by part (a),

$$\left|f(x) - \frac{1}{4}\right| < \frac{\delta}{12}$$

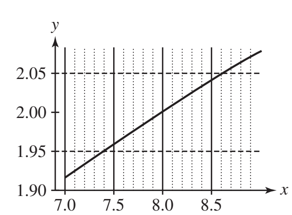
Choosing $\delta = 0.12$ will then guarantee that $|f(x) - \frac{1}{4}| < 0.01$.

(c) Let $\epsilon > 0$ and take $\delta = \min\{1, 12\epsilon\}$. Then, whenever $|x - 2| < \delta$,

$$\left|f(x) - \frac{1}{4}\right| = \left|\frac{1}{x + 2} - \frac{1}{4}\right| = \frac{|2 - x|}{4|x + 2|} \leq \frac{|x - 2|}{12} < \frac{\delta}{12} < \epsilon$$

77. GU Plot the function $f(x) = x^{1/3}$. Use the zoom feature to find a $\delta > 0$ such that if $|x - 8| < \delta$, then $|x^{1/3} - 2| < 0.05$.

SOLUTION The graphs of $y = f(x) = x^{1/3}$ and the horizontal lines $y = 1.95$ and $y = 2.05$ are shown below. From this plot, we see that $\delta = 0.55$ guarantees that whenever $|x - 8| < \delta$, then $|x^{1/3} - 2| < 0.05$.

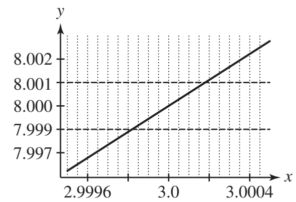


78. Use the fact that $f(x) = 2^x$ is increasing to find a value of δ such that $|2^x - 8| < 0.001$ if $|x - 2| < \delta$. *Hint:* Find c_1 and c_2 such that $7.999 < f(c_1) < f(c_2) < 8.001$.

SOLUTION From the graph below, we see that

$$7.999 < f(2.99985) < f(3.00015) < 8.001$$

Thus, with $\delta = 0.00015$, it follows that $|2^x - 8| < 0.001$ whenever $0 < |x - 3| < \delta$.



79. Prove rigorously that $\lim_{x\to -1}(4+8x) = -4$.

SOLUTION Let $\epsilon > 0$ and take $\delta = \epsilon/8$. Then, whenever $|x-(-1)| = |x+1| < \delta$,

$$|f(x)-(-4)| = |4+8x+4| = 8|x+1| < 8\delta = \epsilon$$

80. Prove rigorously that $\lim_{x\to 3}(x^2 - x) = 6$.

SOLUTION Let $\epsilon > 0$ and take $\delta = \min\{1, \epsilon/6\}$. Because $\delta \le 1$, $|x-3| < \delta$ guarantees $|x+2| < 6$. Therefore, whenever $|x-3| < \delta$,

$$|f(x)-6| = |x^2 - x - 6| = |x-3|\,|x+2| < 6|x-3| < 6\delta \le \epsilon$$