Solutions Manual For

A Course in Real Analysis

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Contents

Preface

This book contains complete solutions to the exercises in *A Course in Real Analysis*. There are over 1600 problems of varying degrees of difficulty, some involving only straightforward application of results in the text, others requiring a deeper analysis. To derive maximum benefit, the reader is urged to attempt a solution before consulting this manual.

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Chapter 1 Solutions

Section 1.2

1. (a) Since $(-a)+a=0$, uniqueness of the additive inverse of $(-a)$ implies that $-(-a)=a$.

   (b) $\big[(ab)+(-a)b\big]=\big[a+(-a)\big]b=0\cdot b=0$, so uniqueness of the additive inverse implies $-(ab)=(-a)b$. A similar argument works for the second equality.

   (c) By (b) and (a), $(-a)(-b)=-(a(-b))=-(-(ab))=ab$.

   (d) By (b), $(-1)a=1(-a)=-a$.

   (e) By commutativity and associativity of multiplication,

   $$(a/b)(bc)=a(b^{-1}b)c=ac=c(d^{-1}d)a=(c/d)(ad),$$

   hence the first equality follows from 1.2.1(h). For the second equality, by commutativity and associativity of multiplication and 1.2.1(i),

   $$(a/b)(c/d)=(ab^{-1})(cd^{-1})=(ac)(b^{-1}d^{-1})=(ac)(bd)^{-1}=(ac)/(bd).$$

   (f) Using commutativity and associativity of multiplication, the distributive law, and 1.2.1(i),

   $$\begin{aligned}a/b+c/d&=ab^{-1}(dd^{-1})+cd^{-1}(bb^{-1})=ad(b^{-1}d^{-1})+bc(b^{-1}d^{-1})\\&=ad(bd)^{-1}+bc(bd)^{-1}(ad+bc)/(bd).\end{aligned}$$

2. Let $r=m/n$ and $s=p/q$ where $m, n, p, q\in\mathbb{N}$ and $nq\neq 0$. By Exercise 1, $r\pm s=(mq\pm pn)/(nq)$ and $rs=(mp)/(nq)$, which are rational. Since $1/s=(pq^{-1})^{-1}=p^{-1}q=q/p$, r/s is the product of rational numbers hence is rational.

3. If $s:=r/x\in\mathbb{Q}$, then, by Exercise 2, $x=r/s\in\mathbb{Q}$, a contradiction. Therefore, $r/x\in\mathbb{I}$. The remaining parts have similar proofs.

4. (a) By commutativity and associativity of multiplication and the distributive law,

   $$\begin{aligned}(x-y)\sum_{j=1}^{n}x^{n-j}y^{j-1}&=\sum_{j=1}^{n}x^{n-j+1}y^{j-1}-\sum_{j=1}^{n}x^{n-j}y^{j}\\&=\sum_{j=0}^{n-1}x^{n-j}y^{j}-\sum_{j=1}^{n}x^{n-j}y^{j}\\&=x^n-y^n.\end{aligned}$$

(b) Replace y in part (a) by $-y$.

(c) Replace x and y in part (a) by x^{-1} and y^{-1}, respectively.

5. The left side of (a) is $\frac{n-1}{n}\frac{n-2}{n}\cdots\frac{1}{n} = \frac{n!}{n^n}$. For (b),

$$\begin{aligned}(2n)! &= \big[2n(2n-2)(2n-4)\cdots 4\cdot 2\big]\big[(2n-1)(2n-3)\cdots 3\cdot 1\big]\\ &= 2^n\big[n(n-1)(n-2)\cdots 2\cdot 1\big]\big[(2n-1)(2n-3)\cdots 3\cdot 1\big].\end{aligned}$$

6. $$\begin{aligned}\binom{n}{k-1}+\binom{n}{k} &= \frac{n!}{(n-k+1)!(k-1)!}+\frac{n!}{(n-k)!k!}\\ &= \frac{kn!+(n-k+1)n!}{(n-k+1)!k!}\\ &= \binom{n+1}{k}\end{aligned}$$

7. Let a_n denote the difference of the two sides of the equation in (a). Combining fractions in the resulting summation leads to

$$a_n = \sum_{k=0}^{n}\frac{n-2k}{(n+2)(k+1)(n-k+1)}.$$

Making the index change $j = n-k$ results in

$$a_n = \sum_{j=0}^{n}\frac{2j-n}{(n+2)(j+1)(n-j+1)} = -a_n.$$

Therefore, $a_n = 0$. Part (b) is proved similarly.

8. $f(k) = k^3-(k-1)^3 = 3k^2-3k+1$.

Section 1.3

1. (a) If $a>0$ and $b<0$, then $-(ab) = a(-b) > 0$ hence $ab<0$.

   (b) If $a>0$ and $1/a<0$, then $1 = a(1/a) < 0$. The converse is similar.

   (c) Follows from $a/b - c/d = (ad-bc)/bd$.

2. Multiply the given inequalities by x, using (d) of 1.3.2.

3. Part (a) follows from a double application of 1.3.2(d). Part (b) follows from (a) by noting that $-y < -x$ and $0 < -b < -a$. Part (c) follows from (a).

4. If $0<x<y$, then multiplying the inequality by $1/(xy)$ and using (d) of 1.3.2 shows that $1/y < 1/x$. If $x<y<0$, then $0<-y<-x$ hence, by the first part, $1/(-x) < 1/(-y)$ so $1/x > 1/y$.

5. If $-1 < x < y$ or $x < y < -1$, then $(y+1)(x+1) > 0$ hence

$$\frac{y}{y+1} - \frac{x}{x+1} = \frac{y-x}{(y+1)(x+1)} > 0.$$

If $x < -1 < y$, then $(y+1)(x+1) < 0$ and the inequality is reversed.

6. (a) By Exercise 1.2.4, $y^n - x^n = (y-x)\sum_{j=1}^{n} y^{n-j}x^{j-1}$. Each term of the sum is positive and less than $y^{n-j}y^{j-1} = y^{n-1}$. Since there are n terms, part (a) follows.

   (b) The inequality is equivalent to

$$n(n+1)xy + ny + nx + x + 1 < n(n+1)xy + ny + nx + y + 1,$$

   which reduces to $x < y$.

7. The given inequality implies that $mx > nx - n$ and $m < n$. Therefore, $n > (n-m)x > x$.

8. $a = ta + (1-t)a < tb + (1-t)b = b$.

9. If the inequality holds, take $x = y = 1$ to get $a \geq -2$ and $x = 1$, $y = -1$ to get $a \leq 2$. Conversely, suppose that $0 \leq a \leq 2$. The inequality then holds trivially if $xy \geq 0$, and if $xy < 0$ then $x^2 + y^2 + axy = (x+y)^2 + (2-a)(-xy) \geq 0$. A similar argument works for the case $-2 \leq a \leq 0$.

10. If $a > b$ then $x := (a-b)/2 > 0$ and $a > b + x$, contradicting the hypothesis.

11. Note that $b > 0$. Suppose $a > b$. Then $x := (1 + a/b)/2 > 1$ and $bx = (a+b)/2 < a$, contradicting the hypothesis.

12. The inequality is equivalent to $a < x^2 + x$ for all $x > 0$. Assume $a > 0$. If $a \geq 1$ then $x = 1/2$ violates the condition. If $0 < a < 1$, then $x := a/4 < 1$ so $a > x + x > x^2 + x$, again, violating the condition. Therefore, $a \leq 0$.

13. (a) Follows from $0 \leq (x-y)^2 = x^2 - 2xy + y^2$.

    (b) $0 \leq (x-y)^2 + (y-z)^2 + (z-x)^2 = 2(x^2+y^2+z^2) - 2(xy+yz+xz)$.

    (c) By expansion, the inequality is equivalent to $2xyzw \leq (yz)^2 + (xw)^2$, which follows from (a).

    (d) Follows from (a).

14. Expand $(x-a)^2 \geq 0$ and divide by x.

15. (a) Write $x - y = (x-z) + (z-y)$ and apply the triangle inequality.

    (b) $|x-L| < \varepsilon$ iff $-\varepsilon < x - L < \varepsilon$.

16. (a) Let $S = \{x_1, \ldots, x_n\}$, where $x_1 < \cdots < x_n$. Then $\min\{S\} = x_1$ and $\max\{-S\} = -x_1$. Part (b) is proved in a similar manner.

    (c) Let $x = \max(S \cup T)$ and assume without loss of generality that $x \in S$. Then $x = \max S$ and $t \le x$ for all $t \in T$ hence $\max T \le x$. Therefore, $x = \max\{\max S, \max T\}$. Part (d) is proved similarly.

17. (a) For the equalities, consider the cases $x \ge 0$ and $x \le 0$.

    (b) Follows from (a).

    (c) Add and subtract the equations $x = y - z$ and $|x| = y + z$.

    (d) Use (b) and the triangle inequality.

    (e) $(x - y)^- = \max\{y - x, 0\} \le y$.

18. If $a \le x \le b$, then $x \le |b|$ and $-x \le -a \le |a|$, hence $|x| \le \max\{|a|, |b|\}$.

19. Consider cases $x \ge y$ and $x \le y$.

20. Set $x := \max\{a, b\}$. By Exercises 16 and 19, $x = \frac{1}{2}(a + b + |a - b|)$ and $\max\{a, b, c\} = \max\{x, c\} = \frac{1}{2}(x + c + |x - c|)$. Substituting the expression for x gives the formula for $\max\{a, b, c\}$. The corresponding formula for $\min\{a, b, c\}$ may be found similarly or may be derived from (a).

21. Assume without loss of generality that $S_1 = S \setminus \{a_1, \ldots, a_k\}$, so $\min S_1 = a_{k+1}$. Each of the remaining sets S_j contains at least one of $a_1, \ldots, a_k$ hence $\min S_j \le a_k < a_{k+1}$, verifying the assertion.

Section 1.4

1. $x \in -A \Rightarrow -x \in A \Rightarrow -x \le \sup A \Rightarrow x \ge -\sup A$. Therefore, $-\sup A$ is a lower bound for $-A$ hence $-\sup A \le \inf(-A)$. Similarly, $a \in A \Rightarrow -a \in -A \Rightarrow -a \ge \inf(-A) \Rightarrow a \le -\inf(-A)$, so $-\inf(-A)$ is an upper bound for A hence $\sup A \le -\inf(-A)$ or $-\sup A \ge \inf(-A)$

2. (a) $\sup = 12, \quad \inf = -12$. (b) $\sup = 1, \quad \inf = -1$.
   (c) $\sup = 3/2, \quad \inf = -3/2$. (d) $\sup = 0, \quad \inf = -2$.

3. (a) $\sup = 3$, $\inf = 2$, (b) $\sup = 3$, $\inf = -2$.
   (c) $\sup = 10/3,\ \inf = 3$. (d) $\sup = \dfrac{3 + \sqrt{5}}{2},\ \inf = -\infty$.
   (e) $\sup = +\infty,\ \inf = -\infty$. (f) $\sup = 2,\ \inf = 3/2$.
   (g) $\sup = \dfrac{3 + \sqrt{2}}{2},\ \inf = \dfrac{3 - \sqrt{2}}{2}$. (h) $\sup = 3,\ \inf = 0$.
   (i) $\sup = \dfrac{1}{2} + \dfrac{\sqrt{2}}{4},\ \inf = \dfrac{1}{2} - \dfrac{\sqrt{2}}{4}$. (j) $\sup = \dfrac{1}{2} + \dfrac{\sqrt{6}}{4},\ \inf = -1/8$.
   (k) $\sup = 4,\ \inf = -2$. (l) $\sup = 2,\ \inf = -2$.
   (m) $\sup = 4/3,\ \inf = -1$. (n) $\sup = 3/2,\ \inf = -5/4$.

4. If B is bounded above then any upper bound of B is an upper bound of A hence $\sup A \le \sup B$. The inequality still holds if B is unbounded above. A similar argument establishes the other inequality.

5. Let $x,\, y \in A$. Then $\pm(x-y) \le \sup A - \inf A$ hence $|x-y| \le \sup A - \inf A$. Since $|x|-|y| \le |x-y|$, $|x|-|y| \le \sup A - \inf A$ so $|x| \le \sup A - \inf A + |y|$. Since x was arbitrary, we have $\sup |A| \le \sup A - \inf A + |y|$ hence $\sup |A| - \sup A + \inf A \le |y|$. Since y was arbitrary it follows that $\sup |A| - \sup A + \inf A \le \inf |A|$.

6. (a) $a \in A$ and $b \in B \Rightarrow a + b \le \sup A + \sup B \Rightarrow \sup (A + B) \le \sup A + \sup B$. The infimum case is similar.

   (b) Since $x > 0$, $xa \le x \sup A$ for all $a \in A$, hence $\sup (xA) \le x \sup A$. Replacing x by $1/x$ proves the inequality in the other direction.

   (c) For any $a \in A$ and $b \in B$, $ab \ge \inf A \inf B$, so $\inf AB \ge \inf A \inf B$. If $inf A = 0$, choose a sequence a_n in A with $a_n \to 0$. Fix any $b \in B$. Then $\inf AB \le a_n b \to 0$ so $\inf AB \le \inf A \inf B$ in this case. Now suppose $inf A \ne 0$. Then $ab \ge \inf AB \Rightarrow a \le b^{-1} \inf AB \Rightarrow \inf A \ge b^{-1} \inf AB \Rightarrow b \inf A \ge \inf AB \Rightarrow b \ge [\inf A]^{-1} \inf AB \Rightarrow \inf B \le [\inf A]^{-1} \inf AB \Rightarrow \inf A \inf B \ge \inf AB$.

   (d) $a \in A \Rightarrow a^r \le (\sup A)^r \Rightarrow \sup A^r \le (\sup A)^r$. Also, $a = (a^r)^{1/r} \le \left(\sup A^r\right)^{1/r}$ hence $\sup A \le (\sup A^r)^{1/r}$.

   (e) $a \in A \Rightarrow \inf A \le a \Rightarrow 1/\inf A \ge 1/a \Rightarrow 1/\inf A \ge \sup A^{-1}$. Also, $1/a \le \sup A^{-1} \Rightarrow a \ge 1/(\sup A^{-1}) \Rightarrow \inf A \ge 1/(\sup A^{-1})$, or $1/(\inf A) \le \sup A^{-1}$

7. Let r denote the infimum. By the approximation property for suprema, there exists $x \in A$ such that $\sup A - r < x \le \sup A$. Suppose $x < \sup A$. Choose $y \in A$ such that $x < y \le \sup A$. Then $y - x < r$, a contradiction. Therefore, $\sup A = x \in A$.

8. For all $x,\, y \in A$, $x < y + r$ hence $\sup A \le y + r$ or $\sup A - r \le y$. Therefore, $\sup A - r \le \inf A$ or $\sup A - \inf A \le r$.

9. Let $a < b$ and let $r \in (a - \sqrt{2}, b - \sqrt{2})$ be rational. Then $r + \sqrt{2} \in (a, b)$ is irrational.

10. If $r_1 < \cdots < r_n$ are rationals in (a, b) then there exists a rational in (r_n, b). Therefore, the number of rationals in (a, b) must be infinite. A similar argument applies to irrationals.

11. Choose $n \in \mathbb{N}$ such that $n(b - a) > 1$ and let $m = \lfloor 2^n a \rfloor + 1$. Then $2^n a < m \le 2^n a + 1 < 2^n b$, the last inequality because $2^n > n$. Therefore, $a < m/2^n < b$.

12. (a) If $n := \lfloor x \rfloor = \lfloor -x \rfloor$, then $x - 1 < n \leq x$ and $-x - 1 < n \leq -x$. Adding these inequalities gives $-2 < 2n \leq 0$ so $n = 0$. The converse is trivial.

    (b) If $n := \lfloor x \rfloor = -\lfloor -x \rfloor$, then $x - 1 < n \leq x$ and $x \leq n < x + 1$. This is possible only if $x = n$. The converse is trivial.

    (c) By definition $-x - 1 < \lfloor -x \rfloor \leq -x$.

    (d) Adding $m - x - 1 < \lfloor m - x \rfloor \leq m - x$ to $x - 1 < \lfloor x \rfloor \leq x$ gives $m - 2 < \lfloor x \rfloor + \lfloor m - x \rfloor \leq m$.

13. (a) Let $s = \sum_{j=0}^{n} x_j$ and $t = \sum_{j=0}^{n} \lfloor x_j \rfloor$. Then $s - 1 < \lfloor s \rfloor \leq s$ and $s - (n+1) < t \leq s$. Adding the first inequality to $-s \leq -t < n + 1 - s$ gives $-1 < \lfloor s \rfloor - t < n + 1$, hence $0 \leq \lfloor s \rfloor - t \leq n$.

    (b) By (a), $\lfloor s \rfloor - t = k$ for some $k = 0, 1, \ldots n$. By definition of $\lfloor s \rfloor$, $s - 1 < k + t \leq s$.

14. Let $x := \left(b^m\right)^{1/n}$ and $y := \left(b^{1/n}\right)^m$. By definition, x is the unique positive solution of $x^n = b^m$. Since $y^n = \left[\left(b^{1/n}\right)^m\right]^n = \left[\left(b^{1/n}\right)^n\right]^m = b^m$, $x = y$.

15. Use Exercise 1.2.4 with $x = a^{1/n}$ and $y = b^{1/n}$.

16. Use Exercise 15.

17. Let $\ell \leq x \leq u$ for all $x \in A$. By the Archimedean principle, there exist positive integers m and n such that $-m < \ell \leq u < n$. Set $N = \max\{m, n\}$.

18. This follows from 1.4.11.

19. Let $x = a + b\sqrt{2}$ and $y = c + d\sqrt{2}$, a, b, c, $d \in \mathbb{Q}$. Then, for example,

$$xy = (ac + 2bd) + (bc + ad)\sqrt{2} \in \mathbb{Q}(\sqrt{2}) \quad \text{and}$$
$$1/y = (c - d\sqrt{2})(c^2 + 2d^2) \in \mathbb{Q}(\sqrt{2}).$$

    The set $\{x \in \mathbb{Q}(\sqrt{2}) : x^2 < \sqrt{3}\}$ is bounded above but has no least upper bound in $\mathbb{Q}(\sqrt{2})$ hence $\mathbb{Q}(\sqrt{2})$ is not complete.

20. For any $a \in \mathbb{N}$, if $r := \sqrt{n + a} + \sqrt{n} \in \mathbb{Q}$, then squaring both sides of $\sqrt{n + a} = r - \sqrt{n}$ shows that $\sqrt{n} \in \mathbb{Q}$ and hence that $n = j^2$ for some $j \in \mathbb{N}$ (1.4.11). Then $\sqrt{n + a} \in \mathbb{Q}$ hence $n + a = k^2$ for some $k \in \mathbb{N}$. Therefore, $a = k^2 - j^2 = (k - j)(k + j)$. If $a = 11$, then $k - j = 1$ and $j + k = 11$ so $n = 25$. If $a = 21$, then either $k - j = 1$ and $j + k = 21$ or $k - j = 3$ and $j + k = 7$. The first choice leads to $j = 10$ and $n = 100$, and the second to $j = 2$ and $n = 4$.

21. Let $r = (\sqrt{n}+1)(\sqrt{n+p}+1)^{-1}$. If $n = (p-1)^2/4$, then $n+p = (p+1)^2/4$, hence $r \in \mathbb{Q}$. Conversely, let $r \in \mathbb{Q}$. Since

$$r^2(n+p) = 2(r-1)\sqrt{n} + n + (1-r)^2,$$

$\sqrt{n}$ is rational and hence n is a perfect square, say $n = m^2$, $m \in \mathbb{N}$ (1.4.11). Since

$$\sqrt{n+p} = r^{-1}(\sqrt{n}+1) - 1 = r^{-1}(m+1) - 1,$$

$\sqrt{n+p}$ is rational hence $n + p = k^2$ for some $k \in \mathbb{N}$. Therefore $p = k^2 - m^2 = (k-m)(k+m)$. Since p is prime, $k - m = 1$ and $k + m = p$. Thus $m = (p-1)/2$, hence $n = (p-1)^2/4$.

Section 1.5

1. Let $P(n)$ be the assertion that $a < x_n < x_{n+1} < b$. Since $x_1 - a < 1$, $x_1 - a < \sqrt{x_1 - a} < 1$ hence $x_1 = a + (x_1 - a) < a + \sqrt{x_1 - a} = x_2 < b$. Therefore, $P(1)$ holds. Assume $P(n)$ holds. Then

$$0 < \sqrt{x_n - a} < \sqrt{x_{n+1} - a} < 1$$

so $a < a + \sqrt{x_n - a} < a + \sqrt{x_{n+1} - a} < a + 1$, which is $P(n+1)$. A similar argument proves the other inequality.

2. Let $P(n)$ be the statement that a set with n members has a largest and a smallest element. Clearly $P(1)$ and $P(2)$ are true. Let $n \geq 2$ and assume that $P(n)$ holds. If S is a set with $n+1$ members then removing a member a from S produces a set T with n members. Let m be the smallest and M the largest element of T. Then $\min\{m, a\}$ is the smallest and $\max\{M, a\}$ the largest element of S. Therefore $P(n+1)$ holds.

3. Let $f(n)$ denote the sum on the left side of the equation and $g(n)$ the sum on the right. Then $f(1) = 1/2 = g(1)$. Now let $n \geq 1$. Then

$$f(n+1) - f(n) = \sum_{k=1}^{2n+2} \frac{(-1)^{k+1}}{k} - \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} = \frac{1}{2n+1} - \frac{1}{2n+2}$$

$$g(n+1) - g(n) = \sum_{k=n+2}^{2n+2} \frac{1}{k} - \sum_{k=n+1}^{2n} \frac{1}{k} = \frac{1}{2n+2} + \frac{1}{2n+1} - \frac{1}{n+1}.$$

Since the right sides are equal, $f(n) = g(n) \Rightarrow f(n+1) = g(n+1)$.

4. Let $S(n)$ denote the sum on the left side of the equation and $g(n)$ the expression on the right. In each part, one easily checks that $S(1) = g(1)$. Now let $n > 1$ and assume that $S(n-1) = g(n-1)$. Then the last term of the sum $S(n)$ is $S(n) - S(n-1) = S(n) - g(n-1)$. This shows that the induction step $S(n) = g(n)$ holds iff the last term of the sum $S(n)$ is $g(n) - g(n-1)$. For example,

(a) $n = \dfrac{n(n+1)}{2} - \dfrac{(n-1)n}{2}$,

(c) $n^3 = \dfrac{n^2}{4}[(n+1)^2 - (n-1)^2]$,

(f) $\dfrac{1}{\sqrt{n}+\sqrt{n-1}} = \sqrt{n} - \sqrt{n-1}$.

5. $\frac{25}{3}n^3 - \frac{15}{2}n^2 + \frac{1}{6}n$.

6. (a) $\displaystyle\sum_{k=1}^{999} k + \sum_{k=1}^{999} k^2 = \frac{999 \cdot 1000}{2} + \frac{999 \cdot 1000 \cdot 1999}{6} = 333,333,000.$

(b) $\displaystyle\sum_{k=1}^{500}(4k^2-1) = 4\,\frac{500 \cdot 501 \cdot 1001}{6} - 500 = 167,166,500.$

(c) $$\sum_{k=1}^{251}(4k-3)(4k-1) = 16\,\frac{251 \cdot 252 \cdot 503}{6} - 16\,\frac{251 \cdot 252}{2} + 3 \cdot 251.$$
$$= 85,348,785$$

7. For $n \geq 1$, let $Q(n)$ be the statement $P(n-1+n_0)$. Then $Q(1) = P(n_0)$ is true. Assume $Q(n) = P(n-1+n_0)$ is true. Then $Q(n+1) = P(n+n_0)$ is true. By mathematical induction, $Q(n) = P(n-1+n_0)$ is true for all $n \geq 1$, that is, $P(n)$ is true for every $n \geq n_0$.

8. In each case, let $f(n)$ be the left side of the inequality and $g(n)$ the right side, and let $P(n) : f(n) < g(n)$. Let n_0 be the base value of n for which $P(n)$ is true. It is straightforward to check that in each case $f(n_0) < g(n_0)$. Assume $P(n)$ holds for some $n \geq n_0$, so that $f(n)/g(n) < 1$. Then

(a) $\dfrac{f(n+1)}{g(n+1)} = \dfrac{2n+3}{2^{n+1}} = \dfrac{f(n)}{2g(n)} + \dfrac{1}{2^n} < 1.$

(b) $\dfrac{f(n+1)}{g(n+1)} = \dfrac{n^2+2n+1}{2^{n+1}} = \dfrac{f(n)}{2g(n)} + \dfrac{2n+1}{2^{n+1}} < 1$

(c) $\dfrac{f(n+1)}{g(n+1)} = \dfrac{2^{n+1}}{(n+1)!} = \dfrac{2}{n+1}\dfrac{f(n)}{g(n)} < 1.$

(d) $\dfrac{f(n+1)}{g(n+1)} = \dfrac{3^{n+1}}{(n+1)!} = \dfrac{3}{n+1}\dfrac{f(n)}{g(n)} < 1.$

(e) $\dfrac{f(n+1)}{g(n+1)} = \dfrac{2^{n+1}(n+1)!}{(n+1)^{n+1}} = \dfrac{f(n)}{g(n)}\dfrac{2}{(1+1/n)^n} < 1.$

(f) $\dfrac{f(n+1)}{g(n+1)} = \dfrac{8^{n+1}(n+1)!}{(2n+2)!} = \dfrac{f(n)}{g(n)}\dfrac{4}{2n+1} < 1.$

9. Check that $6 < \ln(6!)$. For the induction step, use $(n+1)! = (n+1)n!$.

10. The inequality clearly holds for $n = 0$. Suppose $(1+x)^n \ge 1 + nx$ for some $n \ge 0$. Then $(1+x)^{n+1} = (1+x)^n(1+x) \ge (1+nx)(1+x) = 1 + (n+1)x + nx^2 \ge 1 + (n+1)x$.

11. For $n \ge 1$, let $Q(n)$ be the statement that $P(k-1+n_0)$ is true for $k = 1, \ldots, n$. Then $Q(1) = P(n_0)$ is true. Assume $Q(n)$ is true, so $P(k-1+n_0)$ is true for $k = 1, \ldots, n$, equivalently, $P(j)$ is true for $n_0 \le j \le n-1+n_0$. By hypothesis, $P(n+n_0)$ is true hence $P(j)$ is true for $n_0 \le j \le n+n_0$. Thus $P(k-1+n_0)$ is true for $k = 1, \ldots n+1$, that is, $Q(n+1)$ is true. By mathematical induction, $Q(n)$ is true for every $n \ge 1$ hence $P(n)$ is true for every $n \ge n_0$.

12. Obvious for $n = 2$. Let $n > 2$ and suppose the prime factorization holds for all integers m with $2 \le m \le n$. If $n+1$ is prime, we're done. Otherwise $n+1 = mk$ where $2 \le m,\ k < n$. By hypothesis, m and k have prime factorizations hence so does the product.

13. Let g_n denote the expression on the right in the assertion. One checks directly that $g_0 = g_1 = 1$. Let $n \ge 2$ and assume that $f_j = g_j$ for all $2 \le j \le n$. Then

$$\begin{aligned}
g_{n+1} - f_{n+1} = g_{n+1} - f_n - f_{n-1} &= g_{n+1} - g_n - g_{n-1} \\
&= \frac{1}{\sqrt{5}}\left(a^{n+2} - a^{n+1} - a^n\right) + \frac{1}{\sqrt{5}}\left(b^{n+2} - b^{n+1} - b^n\right) \\
&= \frac{a^n}{\sqrt{5}}(a^2 - a - 1) + \frac{b^n}{\sqrt{5}}(b^2 - b - 1) = 0.
\end{aligned}$$

14. Let b_n denote the right side of the equation. One checks directly that $b_n = a_n$ for $n = 0, 1$. Let $n \ge 2$ and assume that $b_j = a_j$ for $2 \le j \le n$. We show that $b_{n+1} = a_{n+1}$ or, equivalently, $2b_{n+1} = b_n + b_{n-1}$:

$$\begin{aligned}
b_n + b_{n-1} &= \left[\frac{(-1)^n}{3 \cdot 2^{n-1}} + \frac{(-1)^{n-1}}{3 \cdot 2^{n-2}}\right](a_0 - a_1) + \frac{2}{3}(a_0 + 2a_1) \\
&= \frac{(-1)^{n-1}(a_0 - a_1)}{3 \cdot 2^{n-2}}\left[\frac{-1}{2} + 1\right] + \frac{2}{3}(a_0 + 2a_1) \\
&= \frac{2(-1)^{n+1}(a_0 - a_1)}{3 \cdot 2^n} + \frac{2}{3}(a_0 + 2a_1) \\
&= 2b_{n+1}.
\end{aligned}$$

15. The set of all nonnegative integers of the form $m - qn$, $q \in \mathbb{Z}$, is nonempty (Archimedean principle) hence has a smallest member $r = m - qn$ (well ordering principle). If $r \ge n$, then $0 \le r - n = m - (q+1)n < r$, contradicting the minimal property of r. Therefore, $m = qn + r$ has the required form. If also $m = q'n + r'$, $q' \in \mathbb{Z}$, $r' \in \{0, \ldots, n-1\}$, then $|q - q'|n = |r - r'| < n$ hence $q' = q$ and $r' = r$.

16. Clearly, $n = 1$ has a decimal representation. Assume all integers $q \leq n$ have decimal representations. By the division algorithm, $n+1 = 10q + d$, $d \in \{0, 1, \ldots, 9\}$. Since $q \leq n$, q has a decimal representation, say $q = d_p d_{p-1} \ldots d_0$. Then

$$n + 1 = \sum_{k=0}^{p} d_k 10^{k+1} + d = d_p d_{p-1} \ldots d_0 d.$$

Therefore, by induction, all positive integers have decimal representations. To see that the representation is unique, suppose that

$$n = \sum_{k=0}^{p} d_k 10^k = \sum_{k=0}^{q} e_k 10^k, \quad d_j,\, e_j \in \{0, 1, \ldots, 9\}.$$

Then

$$e_0 - d_0 = \sum_{k=1}^{p} d_k 10^k - \sum_{k=1}^{q} e_k 10^k,$$

which is divisible by 10. Therefore $e_0 = d_0$ and $\sum_{k=1}^{p} d_k 10^k = \sum_{k=1}^{q} e_k 10^k$. Arguing similarly, we see that $e_1 = d_1$. Continuing in this manner, eventually $p = q$ and $e_j = d_j$, $0 \leq j \leq p$.

Section 1.6

1. $\boldsymbol{x} = \boldsymbol{c} - \dfrac{\boldsymbol{d} \cdot \boldsymbol{e} - (\boldsymbol{b} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d})}{1 - (\boldsymbol{a} \cdot \boldsymbol{b})(\boldsymbol{b} \cdot \boldsymbol{d})}\, \boldsymbol{a}, \quad \boldsymbol{y} = \boldsymbol{e} - \dfrac{\boldsymbol{b} \cdot \boldsymbol{c} - (\boldsymbol{a} \cdot \boldsymbol{b})(\boldsymbol{d} \cdot \boldsymbol{e})}{1 - (\boldsymbol{a} \cdot \boldsymbol{b})(\boldsymbol{b} \cdot \boldsymbol{d})}\, \boldsymbol{d}.$

2. By 1.6.3,

   $$||\boldsymbol{x}+\boldsymbol{y}||_2^2 = ||\boldsymbol{x}||_2^2 + ||\boldsymbol{y}||_2^2 + 2(\boldsymbol{x} \cdot \boldsymbol{y}) \text{ and } ||\boldsymbol{x}-\boldsymbol{y}||_2^2 = ||\boldsymbol{x}||_2^2 + ||\boldsymbol{y}||_2^2 - 2(\boldsymbol{x} \cdot \boldsymbol{y}).$$

   Adding and subtracting gives (a) and (b).

   (c) By the triangle inequality,

   $$||\boldsymbol{x}||_2 = ||\boldsymbol{x} - \boldsymbol{y} + \boldsymbol{y}||_2 \leq ||\boldsymbol{x} - \boldsymbol{y}||_2 + ||\boldsymbol{y}||_2$$

   hence $||\boldsymbol{x}||_2 - ||\boldsymbol{y}||_2 \leq ||\boldsymbol{x} - \boldsymbol{y}||_2$. Similarly, $||\boldsymbol{y}||_2 - ||\boldsymbol{x}||_2 \leq ||\boldsymbol{x} - \boldsymbol{y}||_2$.

   (d) Use induction.

3. By 1.6.3, $||\boldsymbol{x}_1 + \boldsymbol{x}_2 + \cdots + \boldsymbol{x}_k||_2^2 = \displaystyle\sum_{i,j=1}^{n} \boldsymbol{x}_i \cdot \boldsymbol{x}_j = \sum_{j=1}^{k} \boldsymbol{x}_j \cdot \boldsymbol{x}_j.$

4. For the triangle inequality, we have

   $$||\boldsymbol{x} + \boldsymbol{y}||_1 = \sum_{j=0}^{n} |x_j + y_j| \leq \sum_{j=0}^{k} |x_j| + |y_j| = ||\boldsymbol{x}||_1 + ||\boldsymbol{y}||_1$$

and

$$\begin{aligned}
||\boldsymbol{x}+\boldsymbol{y}||_\infty &= \max\{|x_1+y_1|,\dots,|x_n+y_n|\}\\
&\le \max\{|x_1|+|y_1|,\dots,|x_n|+|y_n|\}\\
&\le \max\{|x_1|,\dots,|x_n|\}+\max\{|y_1|,\dots,|y_n|\}\\
&= ||\boldsymbol{x}||_\infty + ||\boldsymbol{y}||_\infty
\end{aligned}$$

The remaining properties are clear.

5. If $||\boldsymbol{x}||_2$, $||\boldsymbol{y}||_2 \le r$ and $0 \le t \le 1$, then, by 1.6.4,

$$||t\boldsymbol{x}+(1-t)\boldsymbol{y}||_2 \le ||t\boldsymbol{x}||_2 + ||(1-t)\boldsymbol{y}||_2 = t||\boldsymbol{x}||_2 + (1-t)||\boldsymbol{y}||_2 \le r.$$

The other sets in the exercise are not convex.

6. $$\begin{aligned}
\left(\sum_{j=1}^n x_j^2\right)^{1/2} &\le \sqrt{n}\max_{j=1,\dots,n}|x_j| \le \sqrt{n}\sum_{j=1}^n |x_j| \le n^{3/2}\max_{j=1,\dots,n}|x_j|\\
&= n^{3/2}\left(\max_{j=1,\dots,n} x_j^2\right)^{1/2} \le n^{3/2}\left(\sum_{j=1}^n x_j^2\right)^{1/2}.
\end{aligned}$$

7. The hypotheses imply that

$$\sum_{j=1}^n x_j^2 = \sum_{j=1}^n y_j^2 = 1 \quad \text{and} \quad \sum_{j=1}^n (x_j+y_j)^2 = 4.$$

It follows that $\sum_{j=1}^n x_jy_j = 1$ and $\sum_{j=1}^n (x_j-y_j)^2 = 0$. The same does not hold for $||\cdot||_\infty$ (take $\boldsymbol{x}=(-1,1)$ and $\boldsymbol{y}=(1,1)$) or for $||\cdot||_1$ (take $\boldsymbol{x}=(1,0)$ and $\boldsymbol{y}=(0,1)$).

8. Use the law of cosines.

9. Direct calculation. For (f) show that $\|\boldsymbol{a}\times\boldsymbol{b}\|^2 = \|\boldsymbol{a}\|^2\,\|\boldsymbol{b}\|^2\cos^2\theta$.

Chapter 2 Solutions

Section 2.1

1. Some possibilities:

   (a) $a_n = [a + b + (-1)^n (b - a)]/2$.

   (b) $a_n = [a + b + (-1)^{\lfloor (n+1)/2 \rfloor}(b - a)]/2$,
   $a_n = \big[a + b + (a - b)\big[\sin(n\pi/2) - \cos(n\pi/2)\big]/2$.

   (c) $a_n = [a + b + (-1)^{\lfloor (n-1)/3 \rfloor}(a - b)]/2$.

   (d) $a_n = \frac{1}{2}(b + c - 2a)x_n^2 + \frac{1}{2}(b - c)x_n + a$, $x_n := \sin\big[(n-1)\pi/2\big]$.

   (e) $a_n = 3 + (-1)^{\lfloor (n+1)/2 \rfloor} + [(-1)^n - 1]/2$.

2. $x_1 = a$, $x_n = a + b - x_{n-1}$, $n > 1$.

3. (a) Since $|(4n-1)/(2n+7) - 2| = 15/(2n+7) < 8/n$, choose any integer $N \ge 8/\varepsilon$.

   (b) If $n \ge 6$, $|(2n^2 - n)/(n^2 + 3) - 2| = |n + 6|/(n^2 + 3) \le 2n/n^2 = 2/n$. Therefore, choose $N \ge \min\{6, 2/\varepsilon\}$.

   (c) $|(5\sqrt{n} + 7)/(3\sqrt{n} + 2) - 5/3| = 11/(9\sqrt{n} + 6) < 11/\sqrt{n}$, so choose any integer $N \ge (11/\varepsilon)^2$.

   (d) For $n \ge 2$, $(n-1)/(\sqrt{n} + 1) \ge (n/2)/2\sqrt{n} = \sqrt{n}/4$, so choose any integer $N \ge 16M^2$.

   (e) $|(2 + 1/n)^3 - 8| = \big[(2 + 1/n)^2 + 2(2 + 1/n) + 4\big]/n \le 19/n$, so choose any integer $N > 19/\varepsilon$.

   (f) $\sqrt{\dfrac{n+2}{n+1}} - 1 = \dfrac{1}{\sqrt{n+1}(\sqrt{n+2} + \sqrt{n+1})} \le \dfrac{1}{n}$, so choose any integer $N > 1/\varepsilon$.

4. The disjoint intervals $(-3/2, -1/2)$ and $(1/2, 3/2)$ each contain infinitely many terms of the sequence. Therefore, no limit can exist.

5. Let $r = pq^{-1}$, p, $q \in \mathbb{Z}$, $q > 0$. For all $n \ge q$, $n!r \in \mathbb{Z}$ hence $\sin(n!r\pi) = 0$.

6. The general term in the sequence may be written $n^{p-1}(1 + n^{-2})^p$, which tends to 1 if $p = 1$, 0 if $p < 1$, and $+\infty$ if $p > 1$.

7. Let $A = \{x_1, \ldots, x_p\}$ and $A_j = \{n : a_n = x_j\}$. One of these sets, say A_1, must have infinitely many members. Since $|x_1 - a| \le |x_1 - a_n| + |a_n - a|$ and $a_n \to a$, letting $n \to +\infty$ through A_1 shows that $x_1 = a$. We may therefore choose $\varepsilon > 0$ so that $I := (a - \varepsilon, a + \varepsilon)$ contains no x_j for $j \ge 2$. Let $N \in \mathbb{N}$ such that $a_n \in I$ for all $n \ge N$. For such n, $a_n = a$.

8. (a) $b_n = (3a_n + 2b_n - 3a_n)/2 \to (c-3a)/2$.

   (b) Let $c_n = 3a_nb_n + 5a_n^2 - 2b_n$. Then

$$b_n = (c_n - 5a_n^2)/(3a_n - 2) \to (1-20)/(6-2) = -19/4.$$

9. (a) 2. (b) $\sqrt{a/b}$. (c) $k/2$. (d) $b/2\sqrt{a}$. (e) 1. (f) $1/2a$. (g) $-ka^{k-1}$. (h) a/k. (i) 0. (j) 0. (k) 1/2. (l) 1.

10. If $|a_n| \le M$ for all n, then $|a_nb_n| \le M|b_n| \to 0$.

11. Use $-r \le a_n - b_n \le r$ and 2.1.4.

12. $\sqrt{n}a_n = (na_n)(1/\sqrt{n}) \to a \cdot 0 = 0$. For a counter example to the converse take $a_n = (-1)^n/n$ or $1/n^{3/4}$.

13. If $a = 0$, given $\varepsilon > 0$ choose N such that $a_n < \varepsilon^k$ for all $n \ge N$. Suppose $a > 0$. Then there exists N such that $a_n > 0$ for all $n \ge N$. By Exercise 1.4.15,

$$|a_n^{1/k} - a^{1/k}| = |a_n - a|\left(\sum_{j=1}^{k} a_n^{1-j/k}a^{(j-1)/k}\right)^{-1} \to 0,$$

since the expression inside the parentheses tends to

$$\sum_{j=1}^{k} a^{1-j/k}a^{(j-1)/k} = ka^{1-1/k} > 0.$$

Therefore, $a_n^{1/k} \to a^{1/k}$.

14. (a) Suppose first that $r > 1$. Set $h_n = r^{1/n} - 1$. Then $h_n > 0$, and by the binomial theorem, $r = (1+h_n)^n > nh_n$. Therefore, by the squeeze principle, $h_n \to 0$. If $r < 1$ consider $1/r$.

    (b) Set $h_n = n^{1/n} - 1$. Then $n = (1+h_n)^n > n(n-1)h_n^2/2$, hence $h_n \to 0$.
    (c) Set $h_n = (r + n^k)^{1/n} - 1$. By the binomial theorem, for $n \ge k$

$$r + n^k = (1+h_n)^n > \frac{n(n-1)\cdots(n-k)h_n^{k+1}}{(k+1)!} > \frac{(n-k)^{k+1}h_n^{k+1}}{(k+1)!},$$

hence $h_n \to 0$.

    (d) Use the inequality $2x/\pi \le \sin x \le x$, $0 \le x \le \pi/2$, and the squeeze principle.

15. Follows from the identities $x = x^+ - x^-$, $x^+ = (|x| + x)/2$, and $x^- = (|x| - x)/2$.

16. Let $s = 1/|r|$ and $h = s - 1$. By the binomial theorem,

$$s^n = (h+1)^n = \sum_{k=0}^{n} \binom{n}{k} h^k.$$

Since $s > 1$, each term in the sum is positive hence, for $n > m$,

$$s^n > \binom{n}{m+1} h^{m+1} = \frac{n(n-1)\cdots(n-m)}{(m+1)!}\, h^{m+1} > \frac{(n-m)^{m+1}}{(m+1)!}\, h^{m+1}.$$

Therefore,

$$0 < |n^m r^n| = \frac{n^m}{s^n} < \frac{n^m (m+1)!}{(n-m)^{m+1} h^{m+1}} = \frac{(m+1)!}{n(1-m/n)^{m+1} h^{m+1}}.$$

Since the term on the right tends to 0 as $n \to +\infty$, the squeeze principle implies that $n^m r^n \to 0$.

17. $a_n < r a_{n-1} < r^2 a_{n-2} < \cdots < r^{n-1} a_1 \to 0$. For the example, take $a_n = 2^{1/n}$.

18. Suppose first that $a \in \mathbb{R}$. Given $\varepsilon > 0$, choose N such that $|a_n - a| < \varepsilon/2$ for all $n > N$. For such n,

$$\begin{aligned}\left|\frac{a_1 + \cdots + a_n}{n} - a\right| &\le \left|\frac{(a_1 - a) + \cdots + (a_N - a)}{n}\right| \\ &\qquad + \left|\frac{(a_{N+1} - a) + \cdots + (a_n - a)}{n}\right| \\ &\le \left|\frac{(a_1 - a) + \cdots + (a_N - a)}{n}\right| + \frac{n-N}{n}\,\frac{\varepsilon}{2}.\end{aligned}$$

The second term on the right in the last inequality is less than $\varepsilon/2$. Also, there exists $N' > N$ such that the first term is less than $\varepsilon/2$ for all $n \ge N'$. For such n, $|(a_1 + \cdots a_n)/n - a| < \varepsilon$.

Now suppose $a_n \to +\infty$. Let $M > 0$ and choose N such that $a_n > 4M$ for all $n > N$. For such n,

$$\begin{aligned}\frac{a_1 + \cdots + a_n}{n} &= \frac{a_1 + \cdots + a_N}{n} + \frac{a_{N+1} + \cdots + a_n}{n} \\ &\ge \frac{a_1 + \cdots + a_N}{n} + \frac{4(n-N)M}{n}.\end{aligned}$$

Choose $N' > N$ such that

$$\frac{n-N}{n} > \frac{1}{2} \quad \text{and} \quad \frac{a_1 + \cdots + a_N}{n} > -M$$

for all $n \ge N'$. For such n, $(a_1 + \cdots + a_n)/n \ge 2M - M = M$.

The converse is false: consider $a_n = (-1)^n$.

19. Choose N such that $a_n - a < \varepsilon$ for all $n \geq N$. For such n,

$$0 \leq \min\{a_1, \ldots, a_n\} - a \leq a_n - a < \varepsilon.$$

Therefore, $\min\{a_1, \ldots, a_n\} \to a$. The converse is false: consider $a_n = 1 + (-1)^n$.

20. Given $\varepsilon > 0$, choose N such that $|a_n|/n < \varepsilon$ for all $n \geq N$. Then

$$b_n := n^{-1} \max\{a_1, \ldots, a_n\} = \max\{\alpha_n, \beta_n\},$$

where

$$\alpha_n := n^{-1} \max\{a_1, \ldots, a_N\}, \quad \beta_n = n^{-1} \max\{a_{N+1}, \ldots, a_n\}$$

Choose $N' > N$ such that $|\alpha_n| < \varepsilon$ for all $n \geq N'$. For such n we also have $-\varepsilon < \beta_n < \varepsilon$, hence $-\varepsilon < b_n < \varepsilon$.

If $\{a_n\}$ is bounded below by c then

$$c/n \leq a_n/n \leq \max\{a_1, \ldots, a_n\}/n.$$

Hence if $(1/n) \max\{a_1, \ldots, a_n\} \to 0$, then $a_n/n \to 0$. The example $a_n = 1 - n$ shows that the converse is not generally true.

21. $(x_1^n + \cdots + x_k^n)^{1/n} = x_k \big[(x_1/x_k)^n + \cdots + (x_{k-1}/x_k)^n + 1\big]^{1/n}$ and

$$1 \leq \big[(x_1/x_k)^n + \cdots + (x_{k-1}/x_k)^n + 1\big]^{1/n} \leq k^{1/n} \to 1.$$

22. Suppose that $c \leq f(x) - x \leq d$ for all x, so $c + jx \leq f(jx) \leq djx$. Summing and using Exercise 1.5.4,

$$nc + xn(n+1)/2 \leq \sum_{j=1}^{n} f(jx) \leq nd + xn(n+1)/2$$

hence

$$c/n + x(1 + 1/n)/2 \leq (1/n^2) \sum_{j=1}^{n} f(jx) \leq d/n + x(1 + 1/n)/2.$$

Letting $n \to +\infty$, we obtain (a). Part (b) is proved similarly.

23. Let $c = a_1/a_0$ and $r = -1/2$. By induction, $\dfrac{a_{n+1}}{a_n} = c^{r^n}$ hence

$$a_{n+1} = \left(\frac{a_{n+1}}{a_n}\right) \cdots \left(\frac{a_1}{a_0}\right) a_0 = a_0 c^{1+r+\cdots r^n} \to a_0 c^{1/(1-r)} = a_0^{1/3} a_1^{2/3}$$